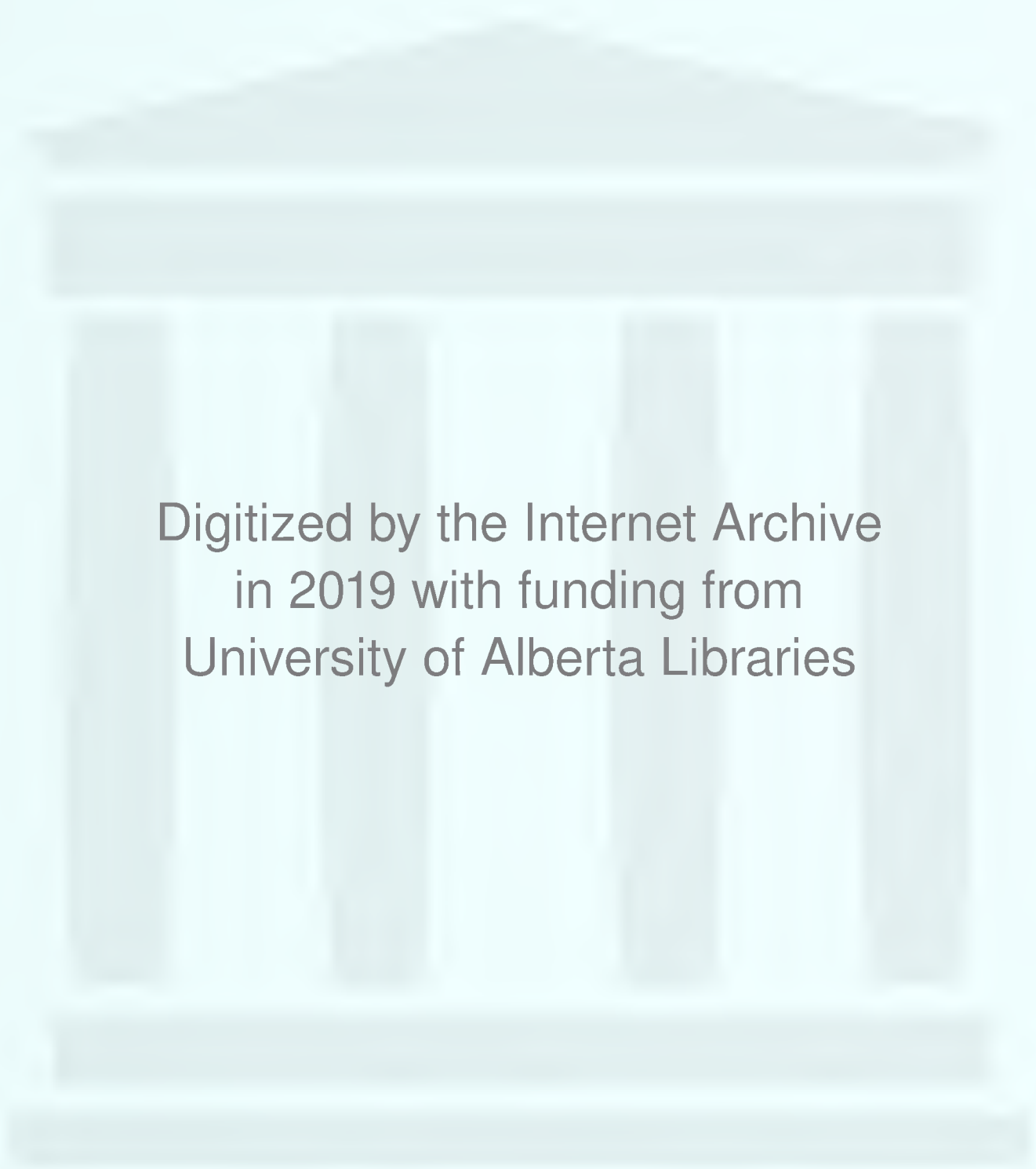




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METHOD OF ANALYSIS OF SHALLOW TRANSLATIONAL SHELLS

by

IYADURAI RAJENDRAM

A THESIS

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FEBRUARY 1965

UNIVERSITY OF ALBERTA
FACULTY OF GRADUATE STUDIES

The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled METHOD OF ANALYSIS OF SHALLOW TRANSLATIONAL SHELLS submitted by Iyadurai Rajendram in partial fulfilment of the requirements for the degree of Master of Science.

ABSTRACT

A numerical method of analysis which takes into account the effect of bending and various boundary conditions for shallow shells of translation is presented in this study. This method is based on developing a physical model that approximates the behaviour of shell structures. The assumptions necessary for developing such a model are similar to those used in deriving the classical bending theories for such shells.

By defining the displacement components in two different ways, two sets of equilibrium equations are obtained. Equations obtained from the models are identical with the corresponding equations obtained by applying central finite difference operators to the differential equations derived by Marguerre and Conrad.

Additional equations and modifications necessary at the boundaries are discussed, including the case where the edges are supported by elastic beams.

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Nomenclature

a, b	=	shell dimensions, semi-spans in the x and y directions
c	=	curvature ratio $= \frac{k_2}{k_1} = \frac{R_1}{R_2}$
D	=	membrane stiffness $= \frac{Et}{(1-\mu^2)}$
E	=	Young's modulus of elasticity.
h_x, h_y	=	shell dimensions, rises in the x and y directions, respectively.
$\vec{i}, \vec{j}, \vec{k}$	=	unit vectors in the x, y and z directions, respectively
K	=	bending stiffness (shell) $= \frac{Et^3}{12(1-\mu^2)}$
L	=	length of the grid
M_{xx}, M_{yy}, M_{xy}	=	stress couples in the shell. (a) moments per unit length for classical theories. (b) moments acting over a length L for the models.
M'_{xx}, M'_{yy}	=	bending moments in the beam about axes in the plane of the shell (model)
M''_{xx}, M''_{yy}	=	bending moments in the beam about axes normal to the shell (model)

$\vec{n}$	= unit vector normal to the shell.
N_{xx}, N_{yy}, N_{xy}	= membrane forces. (a) forces per unit length for classical theories. (b) forces acting over a length L of the shell for the models.
$\bar{N}_{xx}, \bar{N}_{yy}, \bar{N}_{xy}$	= projection of the membrane forces on the x-y plane.
p_n	= component of the load per unit area acting normal to the shell
p_1, p_2	= components of the loads per unit area acting tangential to the shell in the x-z and y-z planes respectively.
p_x, p_y	= components of the loads per unit area in the x and y directions respectively.
Q_x, Q_y	= transverse shears per unit length.
R_1, R_2	= radii of curvatures in the x and y directions, respectively.
r	= distance of any point on the middle surface of the shell from the origin.
S	= shear force in the beam in the plane of the shell (model)
T	= twisting moment in the beam (model)
t	= thickness of the shell
u, v, w	= displacement components in the tangential and normal directions, respectively

$\bar{u}, \bar{v}, \bar{w}$	= displacement components in the x,y and z directions respectively.
V^+, V^-	= transverse shears in the shell (model)
$V^{+'}, V^{-'}$	= transverse shears in the beam (model)
X	= distance of any point in the shell from the origin.
x, y, z	= Cartesian co-ordinates of reference system.
α, β	= angles between the x-y plane and the tangents in the x-z and the y-z planes, respectively.
α_n, β_m	= angles similar to α and β defined for the model (see page 16)
γ_{xy}, γ_{12}	= shearing strains
$\epsilon_x, \epsilon_y, \epsilon_1, \epsilon_2$	= axial strains
μ	= Poisson's ratio
θ	= angle between the tangents in the x-z plane and the y-z plane.
χ_x, χ_y	= changes in curvature in the x and y directions respectively
χ_{xy}	= change in twist
$\lambda_1, \lambda_2, \lambda_3, \lambda_4$	= parameters defining the beam stiffnesses (see pages 66 and 67).
δ	= a small parameter of the order 1/10
τ	= shearing stress.
ξ	= distance of any point in the shell from the middle surface along the normal
∇^2	= $(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})$

CHAPTER I

INTRODUCTION

1.1 General Remarks.

Shells of double curvature, as opposed to shells of single curvature, not only exhibit great strength but also serve as instruments of architectural forms of aesthetic value in roof structures. Among the doubly curved shells, shells of translation permit economical and easier formwork. These special properties of the doubly curved shells of translation have drawn the attention of engineers and architects in recent years. This interest has led to much research on formulation of theories and methods of analysis and to experimental studies on such shells.

1.2 Brief Review of Previous Work.

Membrane theory of shells has been known for quite some time. This theory has been used extensively to analyse doubly curved shells of translation by Reissner [10]^{*}, Parme [9], Candela [2], Flügge [5] and others in recent years. This

^{*}Numbers in brackets [] refer to the list of references.

theory is very attractive due to its simplicity. However, membrane theory imposes restrictions on the type of loadings and edge conditions. Further, this theory will yield inadequate results for shallow shells where the effect of bending will be considerable.

The first bending theory for shallow shells of general shape was published by Marguerre [6] in 1938. Later, in 1944, Vlassov [11] formulated a more general bending theory applicable for all elastic shells using an orthogonal curvilinear co-ordinate system. In 1956, Conrad [4], apparently unaware of the work of Vlassov, developed a similar theory applicable for shallow shells of general shape expressed in Cartesian co-ordinates.

These bending theories result in the necessity of solving eighth order partial differential equations. Conrad [4] attempted a solution for shells with simply supported edges and subjected to concentrated loads by using singular solutions in the form of a rapidly converging series. Apeland [1] used the Levy-type of solution to solve these equations, for which at least two opposite edges must be simply supported. Finite difference technique was used by Noor [8] to study these equations but this technique requires being able to write the differential equations at the boundary and so is restricted to shells with simply

supported, fixed and free edges. At present, no method that permits consideration of a shell supported on all four edges by elastic beams is available in the literature.

1.3 Object and scope.

The object of this study is to develop a method that can be used to analyse doubly curved shallow shells of translation with various boundary conditions and, in particular, shells supported on all four edges by elastic beams. The method that has been attempted in this study is to develop a physical model (analogue) that approximates the behaviour of the shell and then to use this model as a means of analysis.

This study is limited to quadratic surfaces, namely elliptic and hyperbolic paraboloids that are rectangular in plan. The accuracy of this method will be that of the method of finite difference.

CHAPTER II

CLASSICAL BENDING THEORIES

2.1 Equations of the shells.

The present study is restricted to doubly curved shells of translation with rectangular plans as shown in Fig. 2.1. The middle surface of such shells can be represented in Cartesian co-ordinates by the following equation:

$$z(x,y) = \frac{h_x}{a^2} x^2 + \frac{h_y}{b^2} y^2 \quad 2.1$$

To facilitate the classification of the shapes, the above equation can be written as

$$z(x,y) = \frac{x^2}{2R_1} + \frac{y^2}{2R_2} \quad 2.2(a)$$

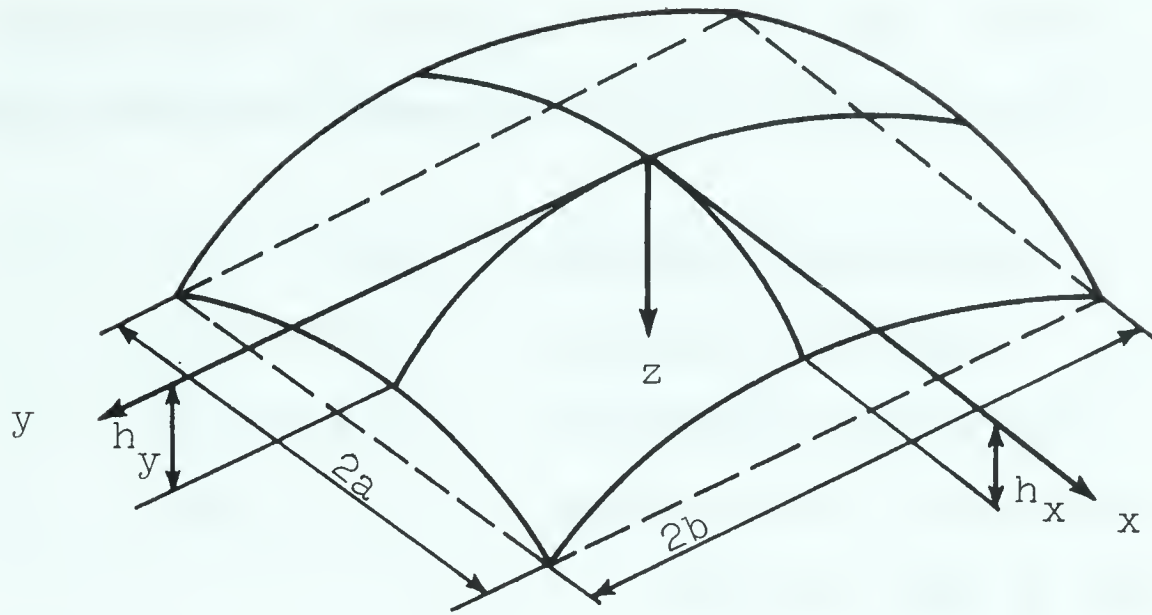
$$= \frac{1}{2R_1} (x^2 + cy^2) \quad 2.2(b)$$

where

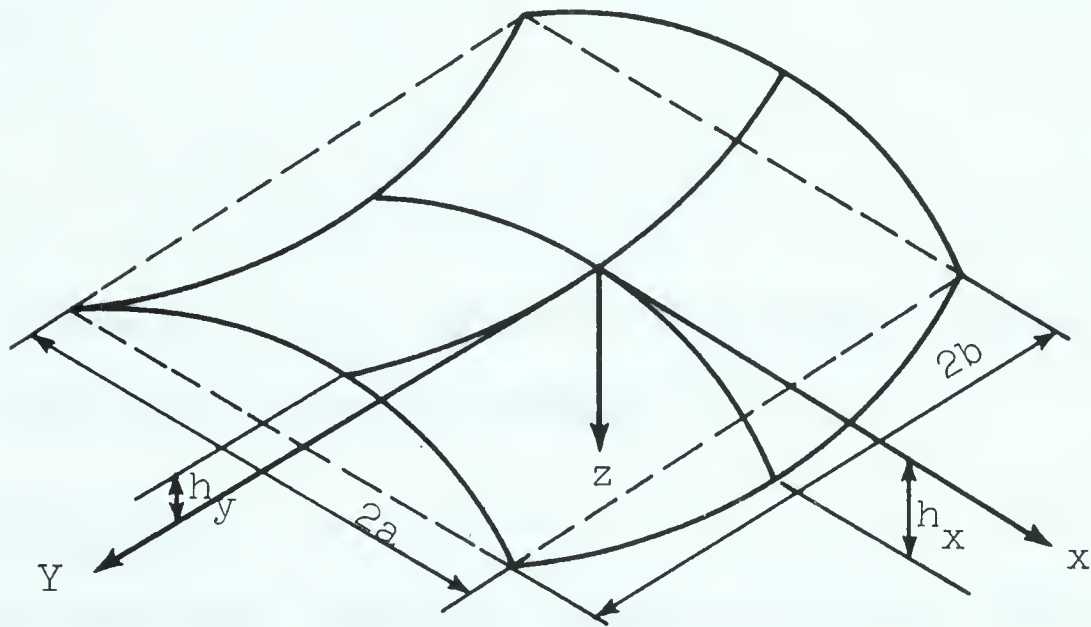
$$R_1 = \frac{a^2}{2h_x}$$

$$R_2 = \frac{b^2}{2h_y}$$

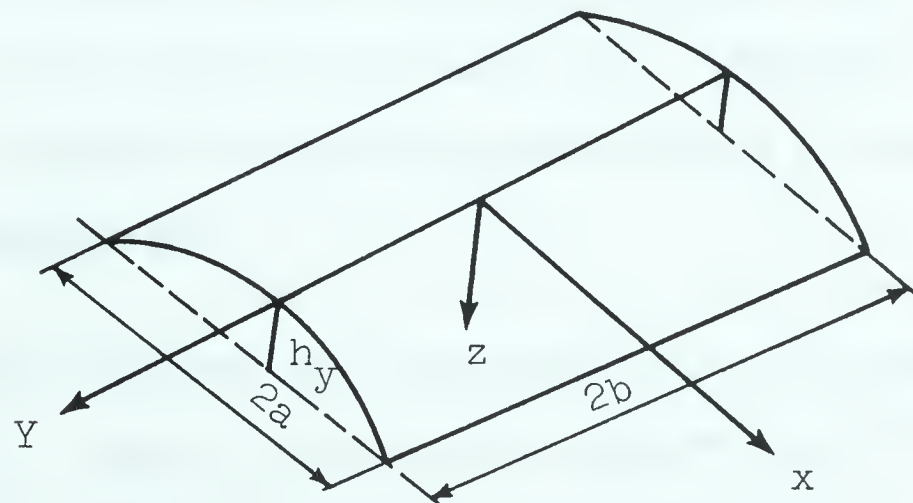
$$c = \frac{R_1}{R_2}$$



(a) Elliptic paraboloid: $c > 0$



(b) Hyperbolic Paraboloid: $c < 0$



(c) Parabolic Cylinder $c = 0$

Fig. 2.1 SHELLS OF TRANSLATION

Depending on the value of c , the Eqn. 2.2(b) represents different shapes of shells as given below

- | | | |
|-----|---------|--------------------------|
| (a) | $c > 0$ | elliptic paraboloid |
| (b) | $c = 0$ | parabolic cylinder |
| (c) | $c < 0$ | hyperbolic paraboloid |
| (d) | $c = 1$ | paraboloid of revolution |
| | | (special case of (a)) |

Eqns. 2.1 and 2.2 can be represented in vector notation by

$$\vec{r} = \vec{i} x + \vec{j} y + \vec{k} z(x,y) \quad 2.3$$

where r = distance of any point on the middle surface of the shell from the origin.

2.2 Assumptions.

The existing classical bending theories are based on the theory of elasticity and hence assume all the limitations of the theory of elasticity. Further, the above theories are limited to shallow shells based on the following additional assumptions

- (i) The squares and the product of the partial derivatives $z_{,x}^*$ and $z_{,y}$ are small compared to unity, i.e.,

*In the rest of the text, the partial derivatives are denoted thus: $z_{,x}$; $u_{,x}$ etc.

$$z_{,x}^2 \ll 1 ; z_{,y}^2 \ll 1 ; z_{,x} \cdot z_{,y} \ll 1 \quad 2.4$$

The above assumption permits one to obtain the following results regarding the geometry of the shell.

(a) curvature in the x direction, $k_1 = z_{,xx} = -\frac{2h}{a^2} = \frac{1}{R_1}$

(b) curvature in the y direction, $k_2 = z_{,yy} = -\frac{2h}{b^2} = \frac{1}{R_2}$

(c) The distance between any two points in the shell surface is approximately equal to the projected length in the x-y plane (horizontal plane).

(ii) The effects of the normal shears Q_x and Q_y on the equilibrium equations in the x and y directions or in the tangential directions are negligible.

(iii) Since the stiffness of the shell is much greater in the plane of the shell than normal to the shell, the magnitudes of the horizontal $(\bar{u}, \bar{v})$ or tangential (u, v) displacements are much smaller than the vertical $(\bar{w})$ or normal (w) displacements. Hence the effects of the displacements in the x and y directions or in the tangential directions on the changes in curvatures and twists will be much smaller and are considered negligible compared to the effect from the normal or vertical displacements. This assumption leads to the following approximations.

$$\left(\frac{u}{w}\right)^2 \ll 1; \left(\frac{u_{,x}}{w_{,x}}\right)^2 \ll 1; \left(\frac{u_{,xx}}{w_{,xx}}\right)^2 \ll 1 \quad 2.5$$

(iv) In the classical theory of shells, it is normally assumed that the thickness, t , of the shell is small compared to the radii of curvatures of the shell and the deflections are small compared to the thickness of the shell. Based on the above assumptions, Conrad [4], in the development of his theory, evaluated the order of magnitudes of various quantities in terms of a parameter δ , a small quantity of the order $1/10$. This facilitated the decision as to which terms could safely be neglected in any given equation.

$$\left. \begin{aligned} z_{,x}, \frac{u}{w}, \frac{u_{,x}}{w_{,x}}, \frac{u_{,xx}}{w_{,xx}} &\sim \delta \\ t z_{,xx} &\sim \delta^2 \end{aligned} \right\} \quad 2.6$$

$$\left. \begin{aligned} \frac{w}{t} &\sim \delta \\ w_{,x} &\sim \delta^2 \\ t w_{,xx} &\sim \delta^3 \\ t^2 w_{,xxx} &\sim \delta^4 \end{aligned} \right\} \quad 2.7$$

2.3 Existing Bending Theories.

A general theory for shallow shells of general shape was first presented by Marguerre. In this theory, the displacements are defined along the directions parallel to the co-ordinate axes. The tangential stress resultants are projected to the x-y plane for the purpose of obtaining the equilibrium equations. The derivation of this theory with a later modification [7] is given in the appendix A and the final equations in terms of the unknown displacements are given below.

$$\begin{aligned}
 & \bar{u}_{,xx} + \frac{(1-\mu)}{2} \bar{u}_{,yy} + \frac{(1+\mu)}{2} \bar{v}_{,xy} + \{z_{,xx} + \frac{(1-\mu)}{2} z_{,yy}\} \bar{w}_{,x} \\
 & + \left\{ \frac{(1+\mu)}{2} z_{,xy} \right\} \bar{w}_{,y} + z_{,x} \bar{w}_{,xx} + \left\{ \frac{(1-\mu)}{2} z_{,x} \right\} \bar{w}_{,yy} \\
 & + \left\{ \frac{(1+\mu)}{2} z_{,y} \right\} \bar{w}_{,xy} + \frac{\bar{p}_x}{D} = 0
 \end{aligned} \tag{2.8}$$

$$\begin{aligned}
 & \bar{v}_{,yy} + \frac{(1-\mu)}{2} \bar{v}_{,xx} + \frac{(1+\mu)}{2} \bar{u}_{,xy} + \{z_{,yy} + \frac{(1-\mu)}{2} z_{,xx}\} \bar{w}_{,y} \\
 & + \left\{ \frac{(1+\mu)}{2} z_{,xy} \right\} \bar{w}_{,x} + z_{,y} \bar{w}_{,yy} + \left\{ \frac{(1-\mu)}{2} z_{,y} \right\} \bar{w}_{,xx} \\
 & + \left\{ \frac{(1+\mu)}{2} z_{,x} \right\} \bar{w}_{,xy} + \frac{\bar{p}_y}{D} = 0
 \end{aligned} \tag{2.9}$$

$$\begin{aligned}
& (\bar{u}_{,x} + z_{,x} \bar{w}_{,x} + \mu \bar{v}_{,y} + \mu z_{,y} \bar{w}_{,y}) z_{,xx} + \\
& (\bar{v}_{,y} + z_{,y} \bar{w}_{,y} + \mu \bar{u}_{,x} + \mu z_{,x} \bar{w}_{,x}) z_{,yy} + \\
& (1-\mu) (\bar{u}_{,y} + \bar{v}_{,x} + z_{,x} \bar{w}_{,y} + z_{,y} \bar{w}_{,x}) z_{,xy} - \\
& \frac{t^2}{12} \nabla^2 \nabla^2 \bar{w} + \frac{p_n}{D} = 0
\end{aligned} \tag{2.10}$$

The stress resultants can be obtained from the displacements by the following relationships

$$N_{xx} = D(\bar{u}_{,x} + z_{,x} \bar{w}_{,x} + \mu \bar{v}_{,y} + \mu z_{,y} \bar{w}_{,y}) \tag{2.11}$$

$$N_{yy} = D(\bar{v}_{,y} + z_{,y} \bar{w}_{,y} + \mu \bar{u}_{,x} + \mu z_{,x} \bar{w}_{,x}) \tag{2.12}$$

$$N_{xy} = D \frac{(1-\mu)}{2} (\bar{u}_{,y} + \bar{v}_{,x} + z_{,y} \bar{w}_{,x} + z_{,x} \bar{w}_{,y}) \tag{2.13}$$

$$M_{xx} = -K (\bar{w}_{,xx} + \mu \bar{w}_{,yy}) \tag{2.14}$$

$$M_{yy} = -K (\bar{w}_{,yy} + \mu \bar{w}_{,xx}) \tag{2.15}$$

$$M_{xy} = -K (1-\mu) \bar{w}_{,xy} \tag{2.16}$$

Vlassov presented a different formulation of the theory for shallow shells of any general shape [11]. In this method, the middle surface of the shell is described in terms of the orthogonal curvilinear co-ordinates and the displacement components are defined along the tangential and normal directions.

Conrad [4], using a technique similar to that of Vlassov, formulated the theory in terms of the Cartesian co-ordinates. In this method, the equation of the shell is expressed in terms of the Cartesian co-ordinates but the displacements are defined along the tangential and normal directions. The equations of equilibrium are obtained by summing the forces in the tangential and normal directions. By co-ordinate transformation, it can be shown that Vlassov's equations are identically the same as Conrad's equations. The derivation of Conrad's equations is shown in Appendix B and the equilibrium equations in terms of the unknown displacements are given below.

$$\begin{aligned}
 u_{,xx} + \frac{(1-\mu)}{2} u_{,yy} + \frac{(1+\mu)}{2} v_{,xy} - \{(k_1 + \mu k_2) w\}_{,x} \\
 - \{(1-\mu) k_{12} w\}_{,y} + \frac{p_1}{D} = 0
 \end{aligned}
 \tag{2.17}$$

$$\begin{aligned}
v_{,yy} + \frac{(1-\mu)}{2} v_{,xx} + \frac{(1+\mu)}{2} u_{,xy} - \{(\mu k_1 + k_2)w\}_{,y} \\
- \{ (1-\mu) k_{12} w \}_{,x} + \frac{p_2}{D} = 0
\end{aligned} \tag{2.18}$$

$$\begin{aligned}
(k_1 + \mu k_2) u_{,x} + (\mu k_1 + k_2) v_{,y} + (1-\mu)k_{12}(u_{,y} + v_{,x}) \\
- \{k_1^2 + 2\mu k_1 k_2 + k_2^2 + 2(1-\mu)k_{12}^2\} w \\
- \frac{t^2}{12} \nabla^2 \nabla^2 w + \frac{p_n}{D} = 0
\end{aligned} \tag{2.19}$$

where $k_{12} = z_{,xy}$.

The stress resultants can be expressed in terms of the displacement components by the following expressions.

$$N_{xx} = D \{u_{,x} + \mu v_{,y} - (k_1 + \mu k_2) w\} \tag{2.20}$$

$$N_{yy} = D \{v_{,y} + \mu u_{,x} - (\mu k_1 + k_2) w\} \tag{2.21}$$

$$N_{xy} = D \frac{(1-\mu)}{2} (u_{,y} + v_{,x} - 2 k_{12} w) \tag{2.22}$$

$$M_{xx} = -K (w_{,xx} + \mu w_{,yy}) \tag{2.23}$$

$$M_{yy} = -K (w_{,yy} + \mu w_{,xx}) \tag{2.24}$$

$$M_{xy} = -K (1-\mu) w_{,xy} \tag{2.25}$$

2.4 Summary.

Even though there are two distinct theories for shallow shells of general shape, Conrad has shown that they are equivalent by introducing Airy's function and reducing the equations to two fourth order equations. The main differences in the two theories given above occurs due to the different methods of defining the displacement components. In one method, the displacement components are defined along the directions parallel to the co-ordinate axes and hence the directions remain the same for the entire shell. In the other method, however, the directions of the displacements are changing from point to point in the shell since the displacements are defined along the tangential and normal directions.

In conclusion, it may be said that Conrad's equations are easier to apply since the coefficients of the equations are independent of the position on the shell surface for shells with quadratic surfaces.

CHAPTER III

METHOD OF ANALYSIS

3.1. Introduction

The technique of using a physical model to approximate the behaviour of elastic structures has been successfully used. An example is the analysis of the complex problems encountered at the boundaries of plates. This concept can be extended to analyze shell structures of arbitrary shapes and with various boundary conditions.

3.2 Description of the physical model.

The physical model is made up of rigid bars, elastic deformable hinges and springs as shown in Fig. 3.1. A line representation of the model is shown in Fig. 3.2. The primary bars form the chords to the shell in planes parallel to the co-ordinate planes and are connected by deformable hinges. The secondary bars are rigidly connected to the mid-points of any two parallel primary bars. Torsion springs lie along the secondary bars and are connected to the primary bars also at the mid-points.

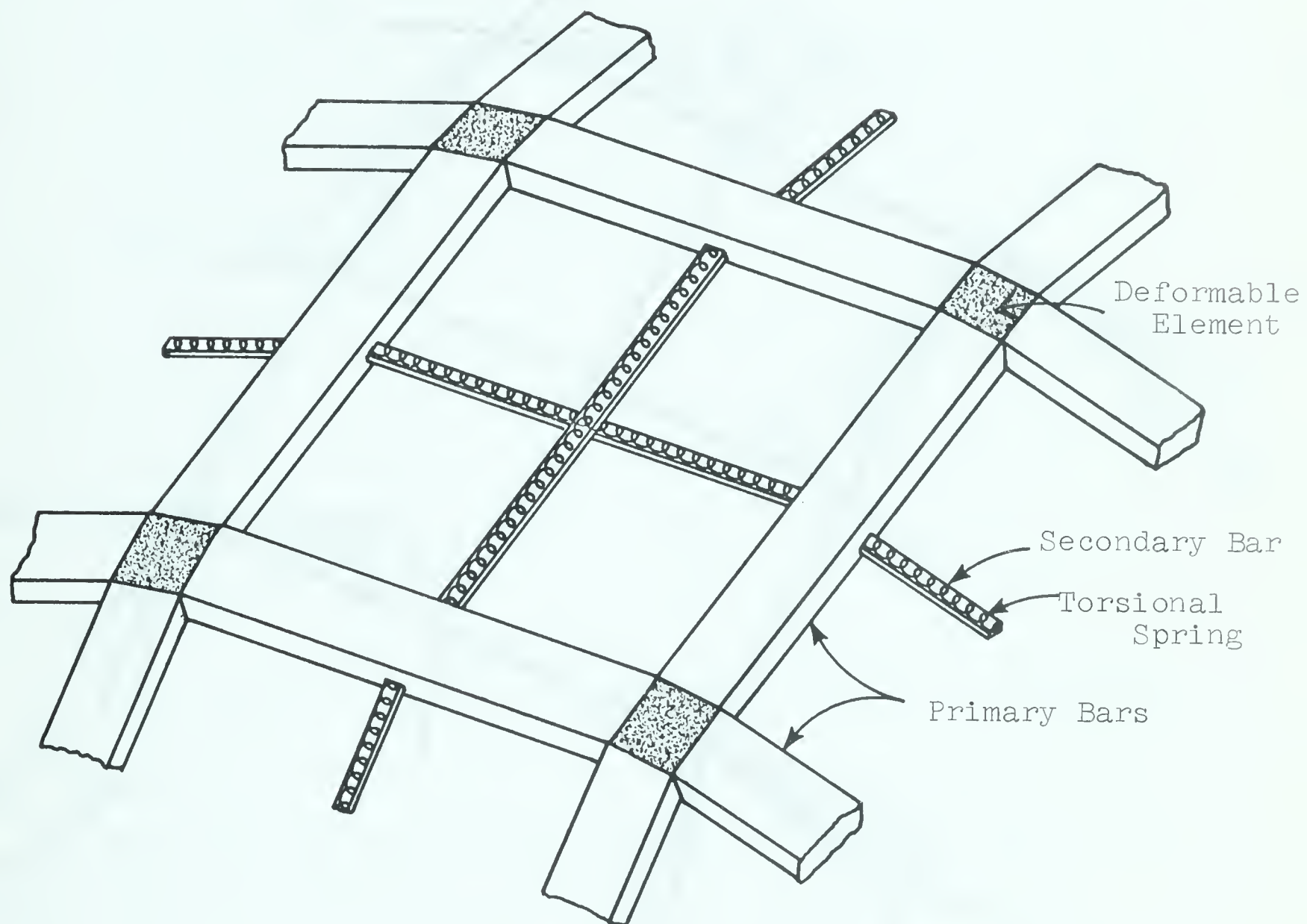


Fig. 3.1 THE PHYSICAL MODEL

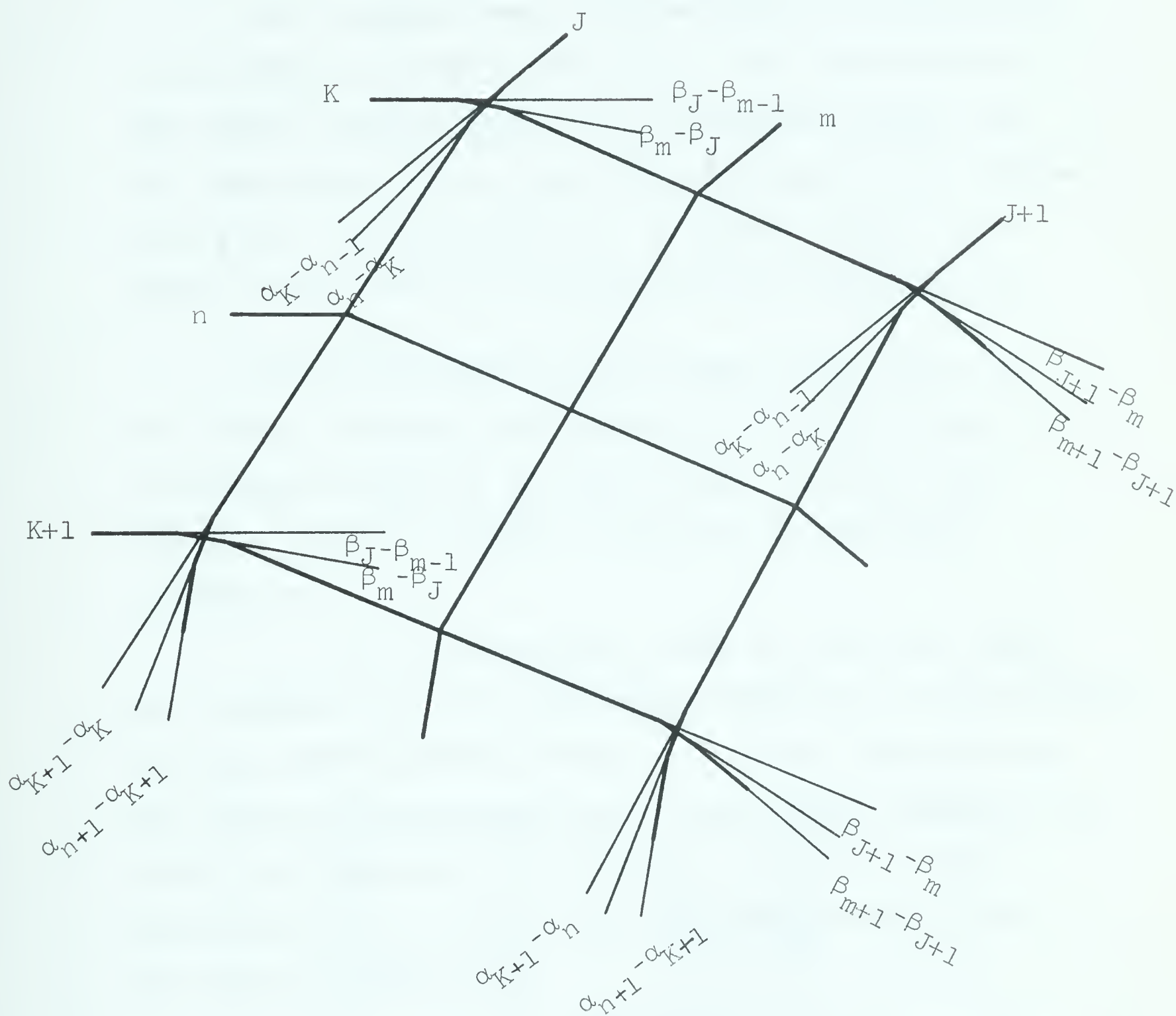


Fig. 3.2 LINE DIAGRAM OF THE MODEL

The notation used for identifying various points in the model is shown in Fig. 3.3. The combination of two capital letters refers to a deformable element and the combination of two small letters refers to the centre of a grid. The combination of a capital letter and a small letter refers to the mid-point of a primary bar.

The construction of the model is such that, in the limit, the model will approach the continuum which is a necessary condition of this concept. Further, the physical model is assumed to possess the following properties:

(i) The external loads and the self weight are assumed to act as concentrated loads and the distribution of the loads is shown in Fig. 3.4. Normal components of the loads are concentrated at the deformable elements, JK, while the tangential or horizontal components in the directions of the primary bars are concentrated at the mid-point of these bars.

(ii) The normal forces, bending moments and vertical shearing forces that act over a length L (grid length) are concentrated at the deformable element and at the ends of the primary bars. Similarly the shearing forces and twisting moments are concentrated at the point where the primary bars and the secondary bars intersect and are defined at the centre of the grid, mn.

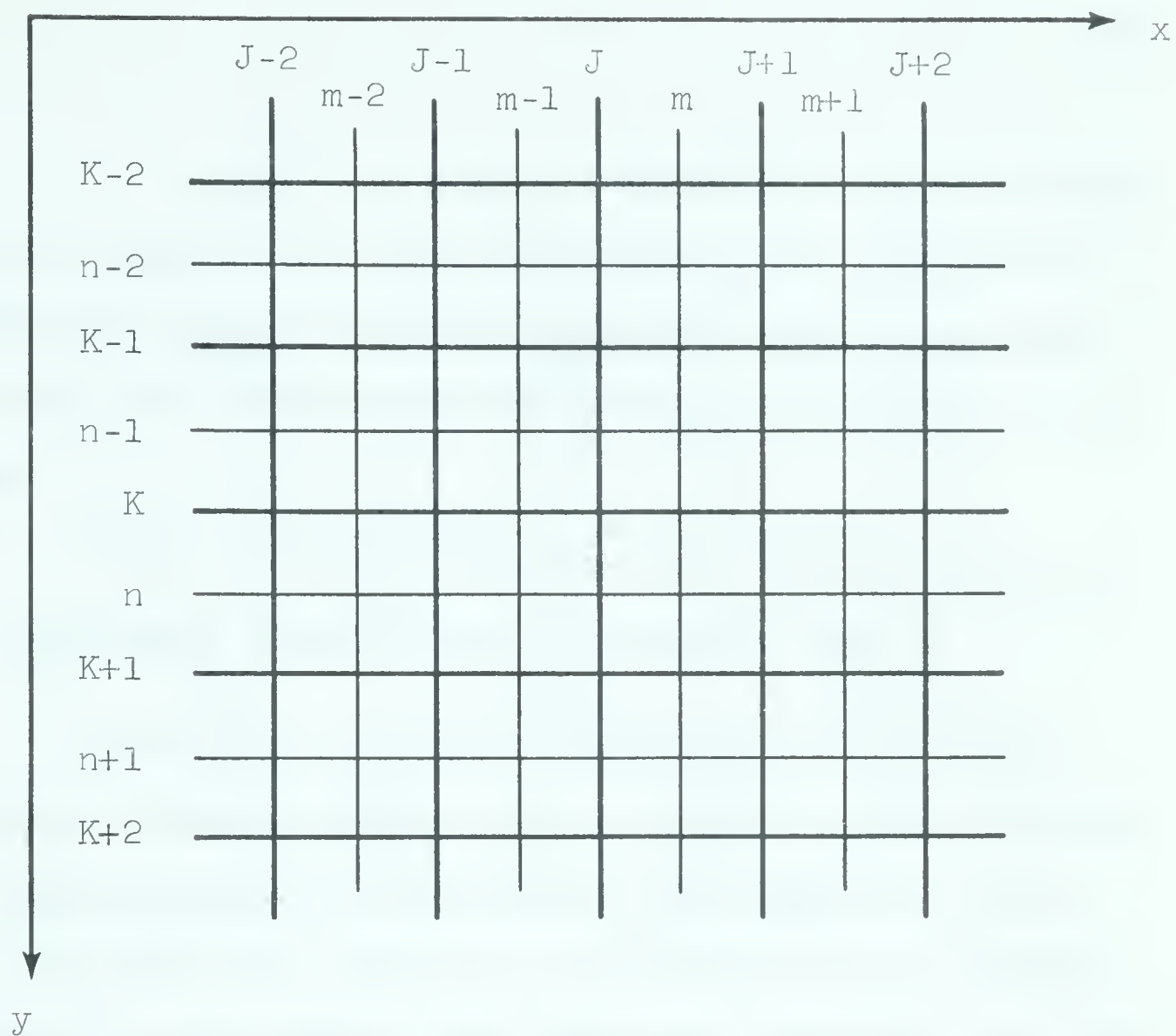


Fig. 3.3 GRID SYSTEM AND NOTATION

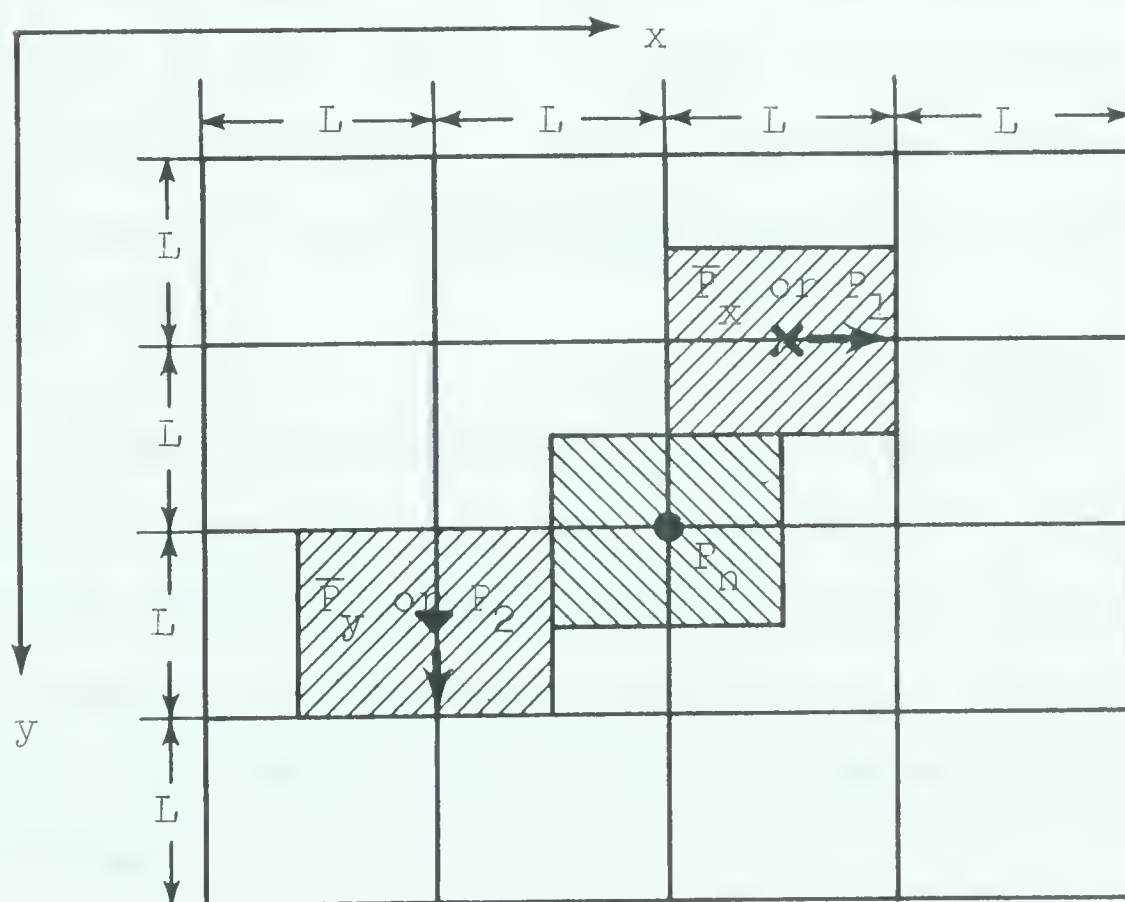


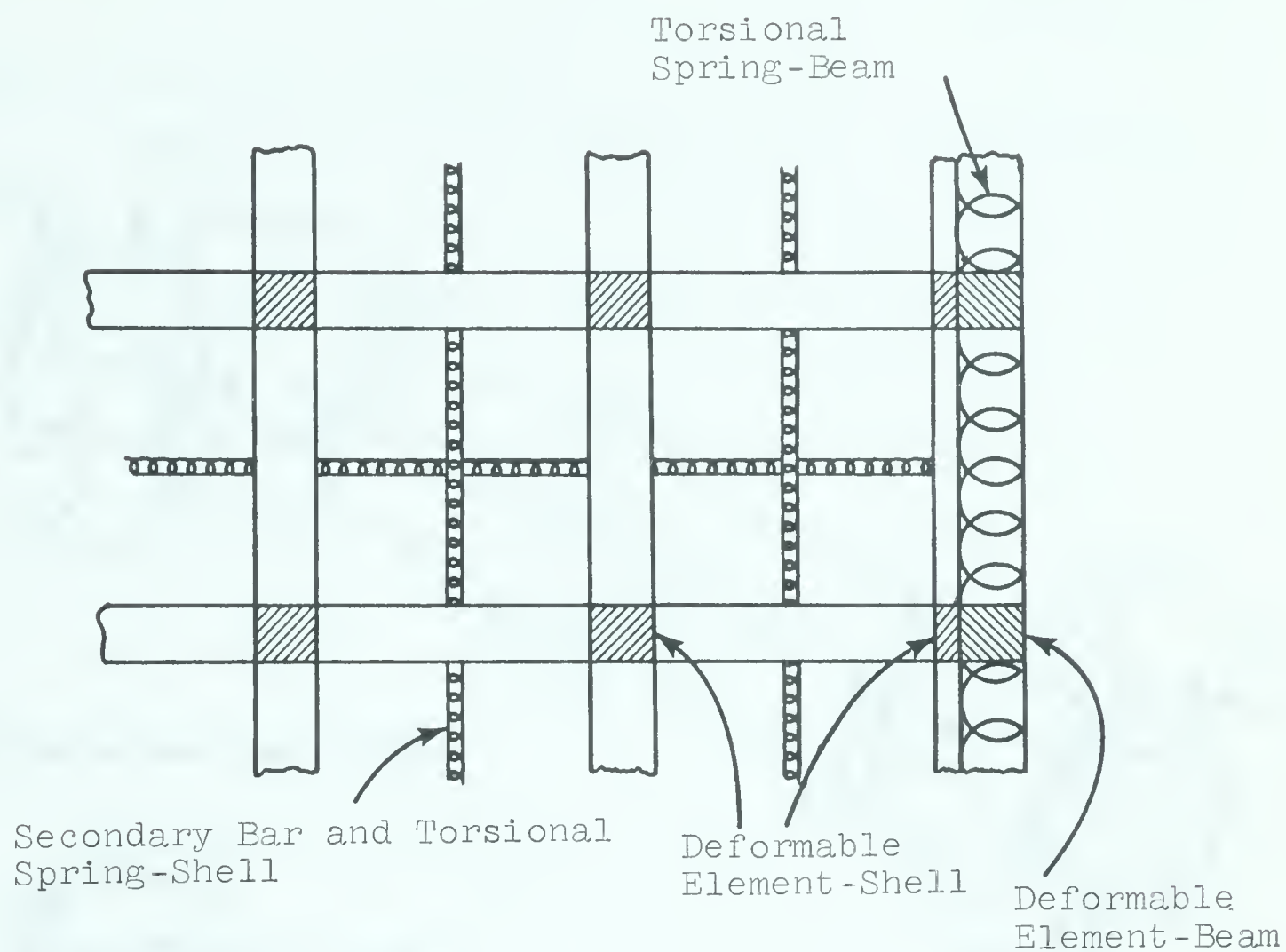
Fig. 3.4 LOAD DISTRIBUTION FOR THE MODELS

(iii) In shells supported by elastic beams, flexural moments and torsional moments are resisted by additional primary bars and torsional springs as shown in Fig. 3.5. These bars and springs form chords to the beam.

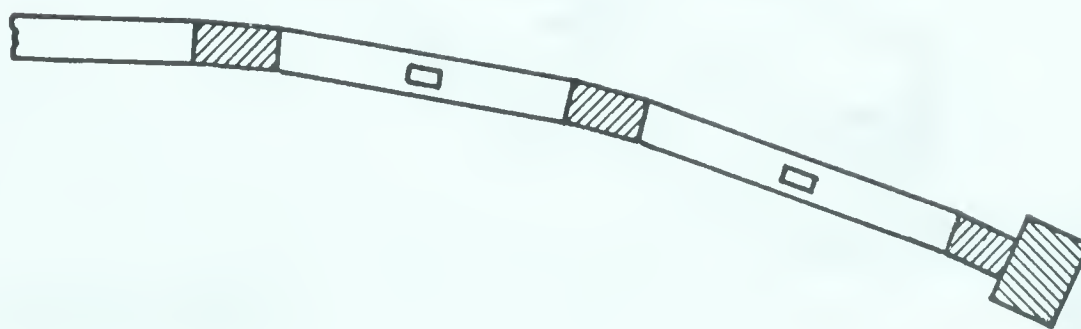
Fig. 3.6 shows the forces and moments acting on the deformable elements and the primary bars.

The types of shells considered in this study possess a special property with respect to their slopes. In planes parallel to x - z plane, the slopes are equal for any particular value of x and similarly, in planes parallel to y - z plane, the slopes are equal for any value of y . In the model, the slope of a deformable element is assumed to be the slope of the shell at that point. The slope of a primary bar is the actual slope of the chord and is equal to the slope of the shell at the mid-point of the grid line.

Using the assumptions given in Chapter II, the lengths of the rigid bars are approximated to the projected lengths in the horizontal planes. For simplicity, only a square grid system is considered in this study. However, this method can be easily extended to a rectangular grid system.



(a) Plan View of the Model Near an Edge Beam



(b) Elevation of Shell-Beam Connection

Fig. 3.5 MODIFICATION OF THE MODEL NEAR THE EDGE BEAM

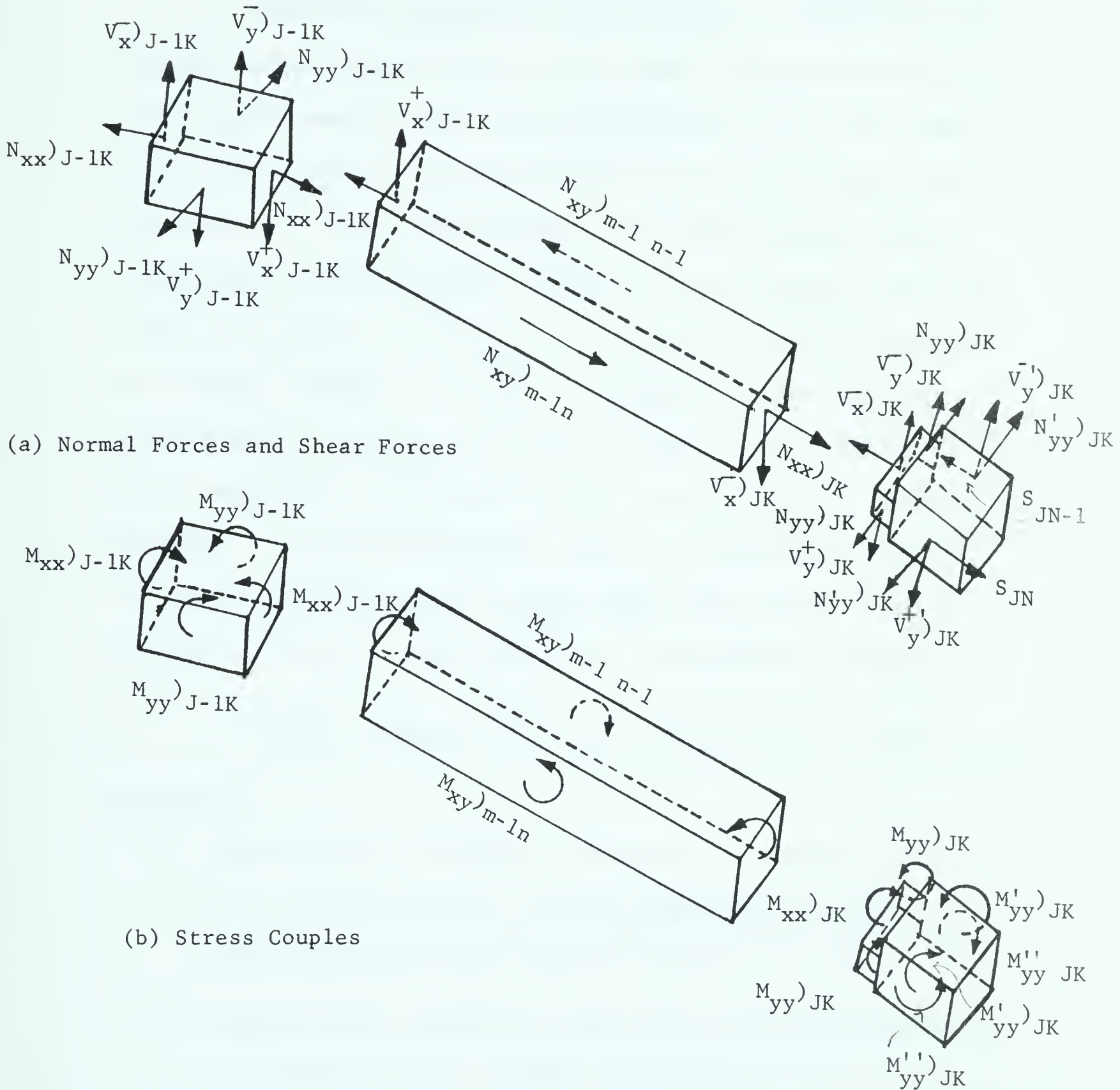


Fig. 3.6 Force System for the Model

3.3 Displacement Components

There are various possibilities of defining the displacement components of the model. Careful choice of the displacement system is necessary so that the model will represent the actual behaviour of the shell. For instance, if the displacements were defined by three components in any three different directions at the same point, it will be found to give results where there will be abrupt changes in forces. However, there are two displacement systems that can be used in the model which will represent the behaviour of the structure. These two systems yield two different sets of equations, one set agrees with Marguerre's theory and the other set with Conrad's theory. The comparison is shown in Chapter IV.

The two displacement systems are as follows:

System A.

- $\bar{u}$: Displacement component defined at the mid-point of a primary bar in a plane parallel to the x-z plane and directed parallel to the x-axis.
- $\bar{v}$: Displacement component defined at the mid-point of a primary bar in a plane parallel to the y-z plane and directed parallel to the y-axis.
- $\bar{w}$: Displacement component defined at the deformable element and directed parallel to the z-axis.

System B.

- u: Displacement component defined at the mid-point of a primary bar in a plane parallel to the x-z plane and directed along the bar.
- v: Displacement component defined at the mid-point of a primary bar in a plane parallel to the y-z plane and directed along the bar.
- w: Displacement component defined at a deformable element and directed along the normal to the shell at that point.

It is noted that the directions of w displacements of the three adjacent deformable elements in a grid line are not strictly parallel due to the change in slope along the grid line. However, in defining the changes in curvatures and twists, these displacements are assumed to be parallel.

3.4 Strain-Displacement Relationships for Model A.

The displacement system used in this model is system A given in section 3.3. This system of displacements is similar to that used by Marguerre.

Consider the extensional strain ϵ_x of the deformable element, JK, in the tangential direction to

the element in the x-z plane. There will be contributions to this strain from $\bar{u}$ and $\bar{w}$ displacement components. These contributions are calculated independently as shown in Fig. 3.7.

$$\text{strain due to } \bar{u} \text{ displacement} = \frac{1}{L} [\bar{u}_{mK} - u_{m-1K}]$$

$$\begin{aligned} \text{strain due to } \bar{w} \text{ displacement} = \frac{1}{2L} [& (\bar{w}_{J+1K} - \bar{w}_{JK}) \tan \beta_m \\ & + (\bar{w}_{JK} - \bar{w}_{J-1K}) \tan \beta_{m-1}] \end{aligned}$$

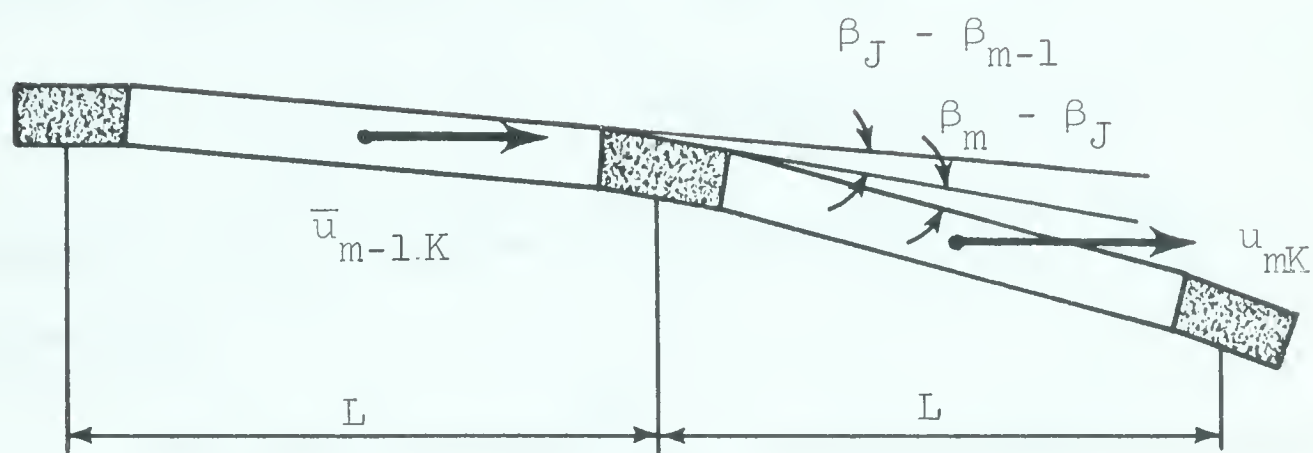
where β_m = angle between the primary bar JK-J+1K and the horizontal plane.

β_{m-1} = angle between the primary bar JK - J-1K and the horizontal plane.

The strain ϵ_x at the point JK is given by

$$\begin{aligned} \epsilon_x)_{JK} = \frac{1}{L}(\bar{u}_{mK} - \bar{u}_{m-1K}) + \frac{1}{2L} [& (\bar{w}_{J+1K} - \bar{w}_{JK}) \tan \beta_m + \\ & (\bar{w}_{JK} - \bar{w}_{J-1K}) \tan \beta_{m-1}] \end{aligned} \quad 3.1$$

Similarly, the strain, ϵ_y , in the y-direction at the point JK is given by

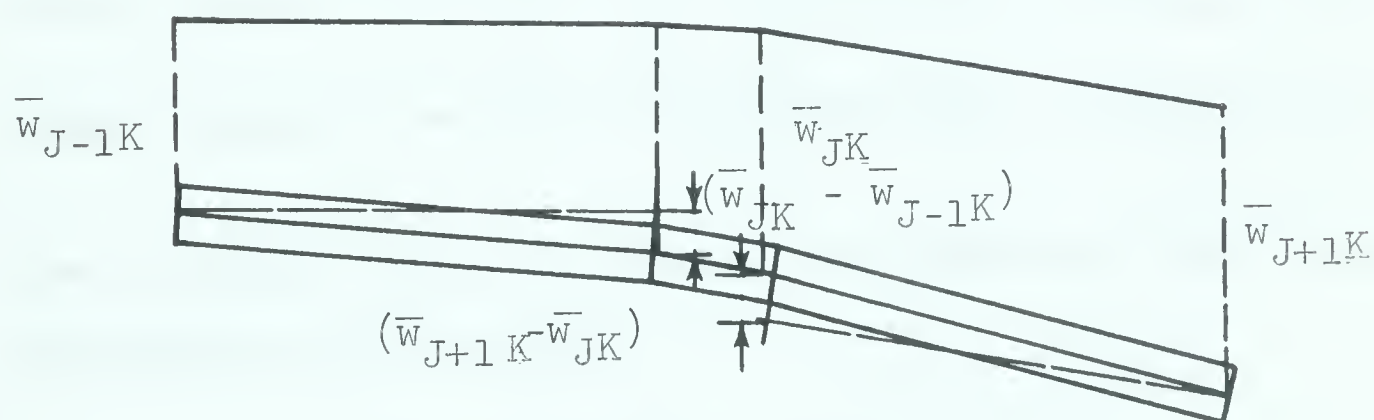


Strain in tangential direction

$$= (\bar{u}_{mK} \cos \beta_J - \bar{u}_{m-1K} \cos \beta_J) \frac{1}{L}$$

$$\approx (\bar{u}_{mK} - \bar{u}_{m-1K}) \frac{1}{L}$$

(a) Strain due to $\bar{u}$ displacement



Strain in the tangential direction

$$= \frac{1}{2L} (\bar{w}_{J+1K} - \bar{w}_{JK}) \tan \beta_m + (\bar{w}_{JK} - \bar{w}_{J-1K}) \tan \beta_{m-1}$$

(b) Strain due to $\bar{w}$ displacement

Fig. 3.7 EFFECT OF DISPLACEMENTS ON NORMAL STRAIN - MODEL A

$$\epsilon_y)_{JK} = \frac{1}{L}(\bar{v}_{Jn} - \bar{v}_{Jn-1}) + \frac{1}{2L} \{ (\bar{w}_{JK+1} - \bar{w}_{JK}) \tan \alpha_n + (\bar{w}_{JK} - \bar{w}_{JK-1}) \tan \alpha_{n-1} \} \quad 3.2$$

where α_n, α_{n-1} = angles similar to β_m, β_{m-1} in the y-direction.

The shear strain, γ_{xy} , is defined as the change in angle between the two intersecting secondary bars. This change in angle is caused by all three displacement components. However, the effect of these displacements is considered independently as shown in Fig. 3.8.

$$\text{Shear strain at mn due to } \bar{u} \text{ displacement} = \frac{1}{L}(\bar{u}_{mK+1} - \bar{u}_{mK})$$

$$\text{Shear strain at mn due to } \bar{v} \text{ displacement} = \frac{1}{L}(\bar{v}_{J+1n} - \bar{v}_{Jn})$$

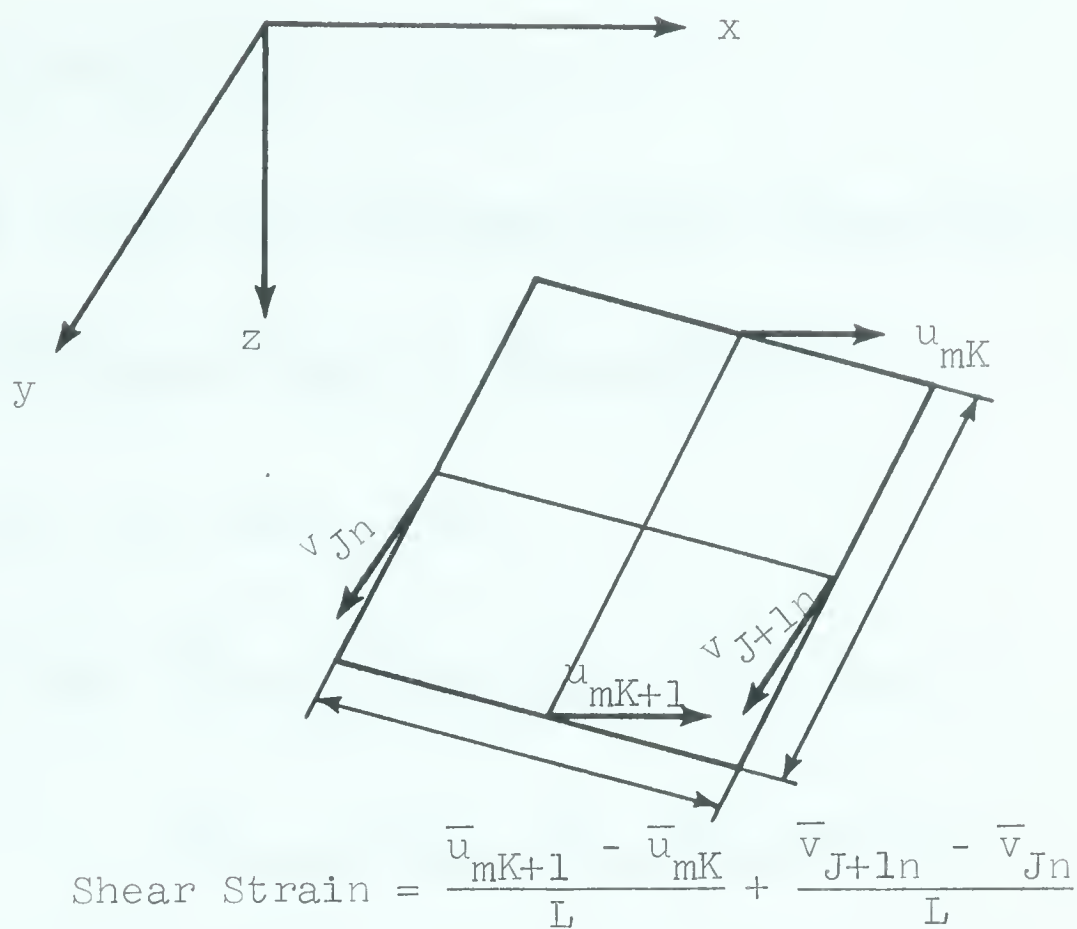
The original angle, θ , between the secondary bars before deformation is given by

$$\cos \theta = \sin \beta_m \sin \alpha_n$$

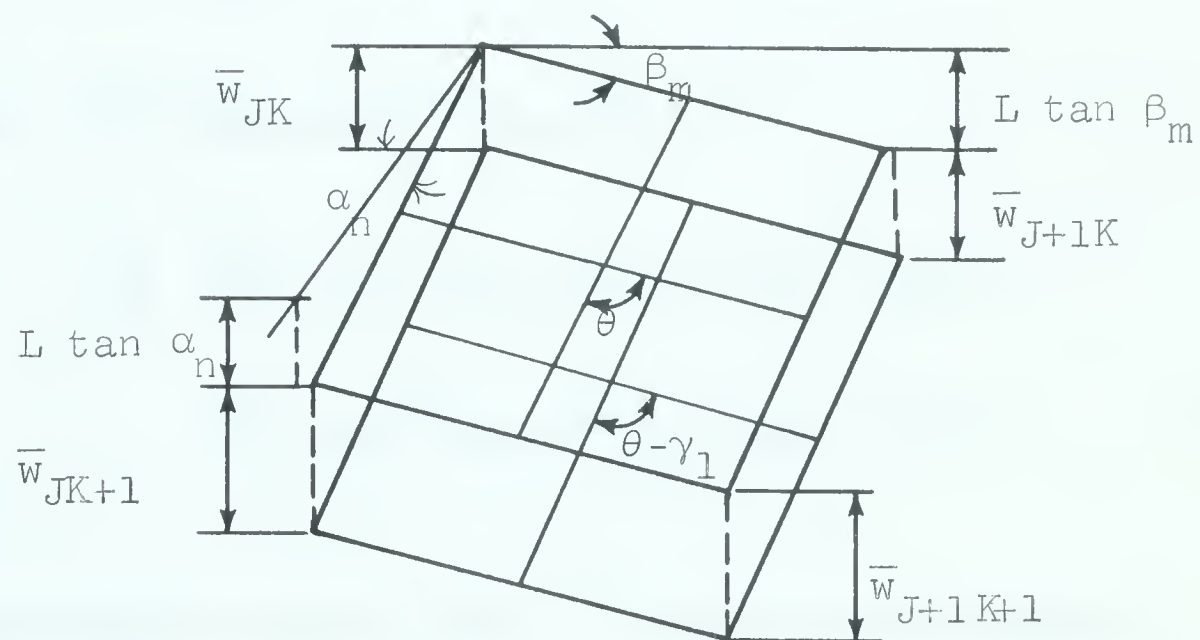
$$\stackrel{\text{r}}{=} \tan \beta_m \tan \alpha_n$$

$$\stackrel{\text{r}}{=} \frac{\pi}{2} - \theta$$

and after $\bar{w}$ displacement is given by



(a) Effect of $\bar{u}$, $\bar{v}$ displacement on Shear Strain



neglecting non-linear terms

$$\gamma_1 = \frac{1}{2L} (\bar{w}_{J+1K+1} + \bar{w}_{J+1K} - \bar{w}_{JK+1} - \bar{w}_{JK}) \tan \alpha_n +$$

$$(\bar{w}_{J+1K+1} + \bar{w}_{JK+1} - \bar{w}_{J+1K} - \bar{w}_{JK}) \tan \beta_m$$

(b) Effect of $\bar{w}$ displacement on shear strain

Fig. 3.8 EFFECT OF DISPLACEMENTS ON SHEAR STRAIN - MODEL A

$$\begin{aligned}
& \cos (\theta - \gamma_1) \doteq \frac{\pi}{2} - \theta + \gamma_1 \\
& = \frac{1}{L} [L \tan \beta_m + \frac{1}{2} \{ (\bar{w}_{J+1K+1} + \bar{w}_{J+1K}) - (\bar{w}_{JK+1} + \bar{w}_{JK}) \}] \frac{1}{L} [L \tan \alpha_n \\
& \quad \frac{1}{2} \{ (\bar{w}_{J+1K+1} + \bar{w}_{JK+1}) - (\bar{w}_{J+1K} - \bar{w}_{JK}) \}]
\end{aligned}$$

Neglecting the non linear terms,

$$\begin{aligned}
\gamma_1 = \frac{1}{2L} \{ (\bar{w}_{J+1K+1} + \bar{w}_{J+1K} - \bar{w}_{JK+1} - \bar{w}_{JK}) \tan \alpha_n + \\
(\bar{w}_{J+1K+1} + \bar{w}_{JK+1} - \bar{w}_{J+1K} - \bar{w}_{JK}) \tan \beta_m \}
\end{aligned}$$

Total shear strain, γ_{xy} , at the point mn is given by

$$\begin{aligned}
\gamma_{xy})_{mn} = \frac{1}{L} (\bar{u}_{mK+1} - \bar{u}_{mK} + \bar{v}_{J+1n} - \bar{v}_{Jn}) + \\
\frac{1}{2L} \{ (\bar{w}_{J+1K+1} + \bar{w}_{J+1K} - \bar{w}_{JK+1} - \bar{w}_{JK}) \tan \alpha_n + \\
(\bar{w}_{J+1K+1} + \bar{w}_{JK+1} - \bar{w}_{J+1K} - \bar{w}_{JK}) \tan \beta_m \} \quad 3.3
\end{aligned}$$

Neglecting the effect of $\bar{u}$, $\bar{v}$ displacement components on the changes in curvatures (χ_x, χ_y) and twist (χ_{xy}) , the following relationships are obtained.

$$\chi_x)_{JK} = \frac{1}{L^2} (\bar{w}_{J+1K} - 2\bar{w}_{JK} + \bar{w}_{J-1K}) \quad 3.4$$

$$\chi_y)_{JK} = \frac{1}{L^2} (\bar{w}_{JK+1} - 2\bar{w}_{JK} + \bar{w}_{JK-1}) \quad 3.5$$

$$\chi_{xy})_{mn} = \frac{1}{L^2} (\bar{w}_{J+1K+1} - \bar{w}_{JK+1} - \bar{w}_{J+1K} + \bar{w}_{JK}) \quad 3.6$$

3.5 Force-Strain Relationships for Model A.

Assuming that the stiffness of the shell of width L (grid length) is concentrated in the deformable element, the following relationships are obtained using Hooke's Law.

$$N_{xx})_{JK} = DL \{ \epsilon_x)_{JK} + \mu \epsilon_y)_{JK} \} \quad 3.7$$

$$N_{yy})_{JK} = DL \{ \epsilon_y)_{JK} + \mu \epsilon_x)_{JK} \} \quad 3.8$$

$$N_{xy})_{mn} = D \left(\frac{1-\mu}{2} \right) L \gamma_{xy})_{mn} \quad 3.9$$

$$M_{xx})_{JK} = -KL \{ \chi_x)_{JK} + \mu \chi_y)_{JK} \} \quad 3.10$$

$$M_{yy})_{JK} = -KL \{ \chi_y)_{JK} + \mu \chi_x)_{JK} \} \quad 3.11$$

$$M_{xy})_{mn} = -KL (1-\mu) \chi_{xy})_{mn} \quad 3.12$$

It is noted that the strains used in the above formulae are not strictly perpendicular to each other. However, it has been assumed to be perpendicular since the error will be small for shallow shells.

3.6 Equilibrium Equations for Model A.

Three equilibrium equations are obtained by summing the forces in the x, y and normal directions for the rigid bars J+1K - JK, JK+1 - JK, and the deformable element JK, respectively.

$$N_{xx})_{J+1K} - N_{xx})_{JK} + N_{xy})_{mn} - N_{xy})_{mn-1} + L^2 \bar{p}_x)_{mK} = 0 \quad 3.13$$

$$N_{yy})_{JK+1} - N_{yy})_{JK} + N_{xy})_{mn} - N_{xy})_{m-1n} + L^2 \bar{p}_y)_{Jn} = 0 \quad 3.14$$

$$\begin{aligned} & V_x^+)_{JK} - V_x^-)_{JK} + V_y^+)_{JK} - V_y^-)_{JK} + \\ & N_{xx})_{JK} \sin(\beta_m - \beta_J) + N_{xx})_{JK} \sin(\beta_J - \beta_{m-1}) \\ & + N_{yy})_{JK} \sin(\alpha_n - \alpha_K) + N_{yy})_{JK} \sin(\alpha_K - \alpha_{n-1}) \\ & + L^2 \bar{p}_n)_{JK} = 0 \end{aligned} \quad 3.15$$

By considering the moment equilibrium of the bar J+1K - JK about an axis through the point J+1K, the vertical shear V_x^+ is obtained as

$$V_x^+)_{JK} = \frac{1}{L} \{ M_{xx})_{J+1K} - M_{xx})_{JK} + M_{xy})_{mn} - M_{xy})_{mn-1} \} \quad 3.16$$

Similarly,

$$V_x^-)_{JK} = \frac{1}{L} \{ M_{xx})_{JK} - M_{xx})_{J-1K} + M_{xy})_{m-1n} - M_{xy})_{m-1n-1} \} \quad 3.17$$

$$V_y^+)_{JK} = \frac{1}{L} \{M_{yy})_{JK+1} - M_{yy})_{JK} + M_{xy})_{mn} - M_{xy})_{m-1n}\} \quad 3.18$$

$$V_y^-)_{JK} = \frac{1}{L} \{M_{yy})_{JK} - M_{yy})_{JK-1} + M_{xy})_{mn-1} - M_{xy})_{m-1n-1}\} \quad 3.19$$

Eqns. 3.1 to 3.3 and 3.13 to 3.15 contain trigonometric terms which can be expressed in terms of the co-ordinates of the various points and shell constants by the following identities.

$$\begin{aligned} \sin (\beta_m - \beta_J) &= \sin \beta_m \cos \beta_J - \cos \beta_m \sin \beta_J \\ &\doteq \tan \beta_m - \tan \beta_J \end{aligned} \quad 3.20(a)$$

Similarly,

$$\sin (\beta_J - \beta_{m-1}) \doteq \tan \beta_J - \tan \beta_{m-1} \quad 3.20(b)$$

$$\sin (\alpha_n - \alpha_K) \doteq \tan \alpha_n - \tan \alpha_K \quad 3.20(c)$$

$$\sin (\alpha_K - \alpha_{n-1}) \doteq \tan \alpha_K - \tan \alpha_{n-1} \quad 3.20(d)$$

Using Eqns. 2.1 and 2.2 the following relationships are obtained.

$$\tan \beta_m = \frac{2h_x}{a} \quad x_m = \frac{x_m}{R_1} \quad 3.21(a)$$

$$\tan \beta_{m-1} = \frac{2h_x}{a} \quad x_{m-1} = \frac{x_{m-1}}{R_1} \quad 3.21(b)$$

$$\tan \alpha_n = \frac{2h_y}{b^2} \quad y_n = \frac{y_n}{R_2} \quad 3.21(c)$$

$$\tan \alpha_{n-1} = \frac{2h_y}{b^2} \quad y_{n-1} = \frac{y_{n-1}}{R_2} \quad 3.21(d)$$

$$\tan \beta_m - \tan \beta_{m-1} = \frac{x_m - x_{m-1}}{R_1} = \frac{L}{R_1} \quad 3.21(e)$$

$$\tan \alpha_n - \tan \alpha_{n-1} = \frac{y_n - y_{n-1}}{R_2} = \frac{L}{R_2} \quad 3.21(f)$$

Using the above relationships, the stress resultants can be expressed in terms of the displacements and the co-ordinates of various points. These expressions are substituted into Eqns. 3.13 to 3.15 to obtain the equilibrium equations in terms of the displacements as follows.

$$\begin{aligned} & \bar{u}_{m+1K} - 2\bar{u}_{mK} + \bar{u}_{m-1K} + \frac{(1-\mu)}{2} (\bar{u}_{mK+1} - 2\bar{u}_{mK} + \bar{u}_{mK-1}) \\ & + \frac{(1+\mu)}{2} (\bar{v}_{J+1n} - \bar{v}_{J+1n-1} - \bar{v}_{Jn} + \bar{v}_{Jn-1}) \\ & + (A_1 x_{m+1}) \bar{w}_{J+2K} + (A_2 - A_3 x_m) \bar{w}_{J+1K} - (A_2 + A_3 x_m) \bar{w}_{JK} \\ & + (A_1 x_{m-1}) \bar{w}_{J-1K} + (A_4 x_m + A_5 y_n) \bar{w}_{J+1K+1} + (A_4 x_m - A_5 y_n) \bar{w}_{JK+1} \\ & + (A_4 x_m - A_5 y_{n-1}) \bar{w}_{J+1K-1} + (A_4 x_m + A_5 y_{n-1}) \bar{w}_{JK-1} \\ & + \bar{p}_x)_{mK} \frac{L^2}{D} = 0 \end{aligned} \quad 3.22$$

Where

$$A_1 = \frac{1}{2R_1}$$

$$A_2 = \left\{ \frac{(1-3\mu)}{4R_2} - \frac{1}{2R_1} \right\} L$$

$$A_3 = \frac{2 - \mu}{2R_1}$$

$$A_4 = \frac{(1-\mu)}{4R_1}$$

$$A_5 = \frac{1 + \mu}{4R_2}$$

$$\begin{aligned} & (\bar{v}_{Jn+1} - 2\bar{v}_{Jn} + \bar{v}_{Jn-1}) + \frac{(1-\mu)}{2} (\bar{v}_{J+1n} - 2\bar{v}_{Jn} + \bar{v}_{J-1n}) \\ & + \frac{(1+\mu)}{2} (\bar{u}_{mK+1} - \bar{u}_{mK} - \bar{u}_{m-1K+1} + \bar{u}_{m-1K}) \\ & + (B_1 y_{n+1}) \bar{w}_{JK+2} + (B_2 - B_3 y_n) \bar{w}_{JK+1} \\ & - (B_2 + B_3 y_n) \bar{w}_{JK} + (B_1 y_{n-1}) \bar{w}_{JK-1} \\ & + (B_4 y_n + B_5 x_m) \bar{w}_{J+1K+1} + (B_4 y_n - B_5 x_m) \bar{w}_{J+1K} \\ & + (B_4 y_n - B_5 x_{m-1}) \bar{w}_{J-1K+1} + (B_4 y_n + B_5 x_{m-1}) \bar{w}_{J-1K} \\ & + \bar{p}_y)_{Jn} \frac{L^2}{D} = 0 \end{aligned}$$

3.23

where

$$B_1 = \frac{1}{2R_2}$$

$$B_2 = \left\{ \frac{(1-3\mu)}{4R_1} - \frac{1}{2R_2} \right\} L$$

$$B_3 = \frac{(2-\mu)}{2R_2}$$

$$B_4 = \frac{(1-\mu)}{4R_2}$$

$$B_5 = \frac{(1+\mu)}{4R_1}$$

$$(20+C_1)\bar{w}_{JK} - (8+C_2x_m)\bar{w}_{J+1K} - (8-C_2x_{m-1})\bar{w}_{J-1K}$$

$$- (8+C_3y_n)\bar{w}_{JK+1} - (8-C_3y_{n-1})\bar{w}_{JK-1}$$

$$+ 2(\bar{w}_{J+1K+1} + \bar{w}_{J-1K+1} + \bar{w}_{J+1K-1} + \bar{w}_{J-1K-1})$$

$$+ 1(\bar{w}_{J+2K} + \bar{w}_{J-2K} + \bar{w}_{JK+2} + \bar{w}_{JK-2})$$

$$- C_4(\bar{u}_{mK} - \bar{u}_{m+K}) - C_5(\bar{v}_{Jn} - \bar{v}_{Jn-1})$$

$$- \frac{L^4}{K} p_n)_{JK} = 0$$

3.24

Where $C_1 = \frac{12L^4}{t^2} \left\{ \frac{1}{2} \left(\frac{1}{R_1^2} + \frac{1}{R_2^2} \right) + \frac{\mu}{R_1 R_2} \right\}$

$$C_2 = \frac{12L^3}{t^2} \left\{ \frac{1}{R_1} + \frac{\mu}{R_2} \right\} \frac{1}{2R_1}$$

$$C_3 = \frac{12L^3}{t^2} \left\{ \frac{\mu}{R_1} + \frac{1}{R_2} \right\} \frac{1}{2R_2}$$

$$C_4 = \frac{12L^3}{t^2} \left\{ \frac{1}{R_1} + \frac{\mu}{R_2} \right\}$$

$$C_5 = \frac{12L^3}{t^2} \left\{ \frac{\mu}{R_1} + \frac{1}{R_2} \right\}$$

Operator form of Eqns. 3.22 to 3.24 are given below.

$$\left\{ \begin{array}{c} \text{Diagram 1: A square element with a central node (circled with an X). Horizontal arrows pointing out from the center are labeled 1. Vertical arrows pointing out from the center are labeled } \frac{(1-\mu)}{2}. \text{ The central node is labeled } -(3-\mu). \end{array} \right\} \bar{u} + \left\{ \begin{array}{c} \text{Diagram 2: A square element with a central node (circled with an X). Horizontal arrows pointing in from the center are labeled } \frac{(1+\mu)}{2}. \text{ Vertical arrows pointing in from the center are labeled } -\frac{(1+\mu)}{2}. \end{array} \right\} \bar{v} +$$

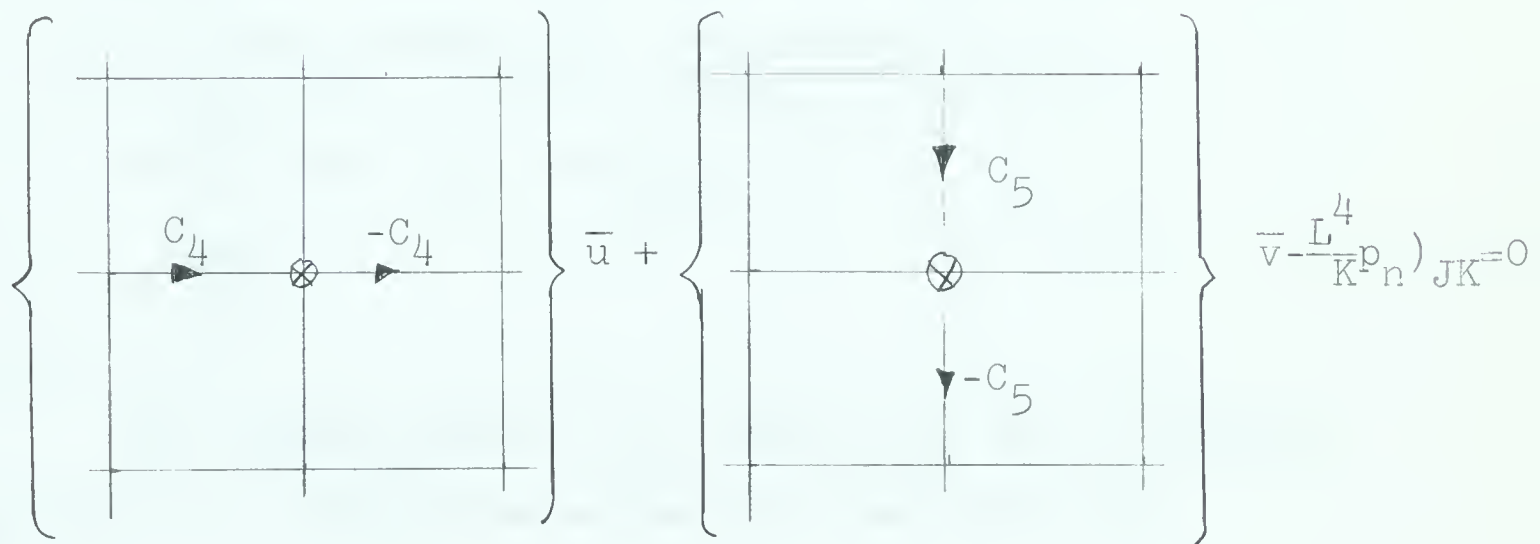
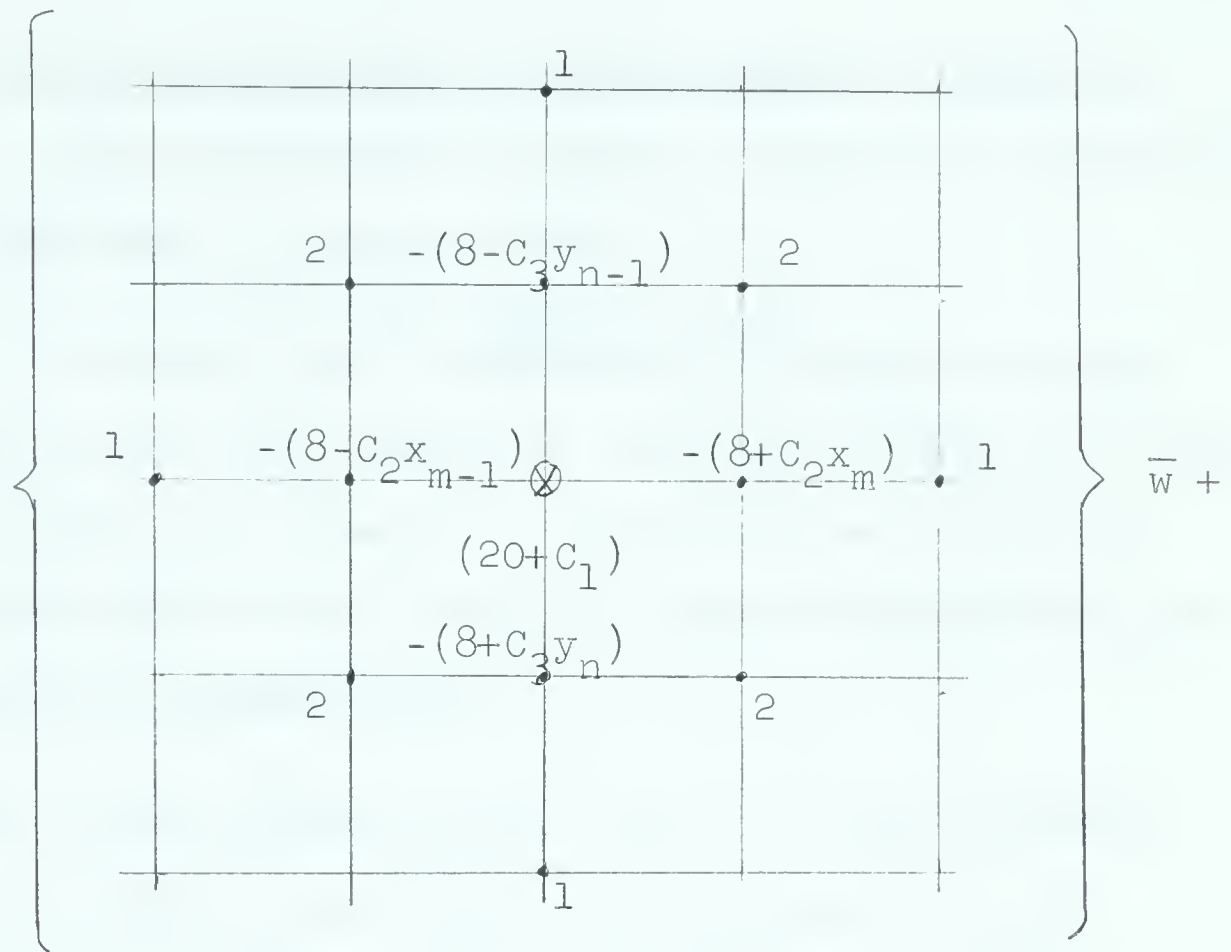
$$\left\{ \begin{array}{c} \text{Diagram 3: A square element with a central node (circled with an X). The central node is labeled } -(A_2 + A_3 x_m). \text{ The four corner nodes are labeled: } (A_1 x_{m-1}), (A_1 x_{m+1}), (A_4 x_m - A_5 y_n), \text{ and } (A_4 x_m + A_5 y_n). \end{array} \right\} \bar{w} + \bar{p}_x \frac{L^2}{D} = 0$$

Equilibrium Equation in the x-direction - Point mK

$$\left\{ \begin{array}{c} \begin{array}{c} \downarrow 1 \\ \hline \frac{(1-\mu)}{2} \quad \downarrow \quad \frac{(1-\mu)}{2} \\ \hline \otimes \\ \hline \downarrow 1 \end{array} \\ \hline \end{array} \right\} \bar{v} + \left\{ \begin{array}{c} \begin{array}{c} \frac{(1+\mu)}{2} \quad \frac{(1+\mu)}{2} \\ \hline \downarrow \quad \downarrow \\ \hline \otimes \\ \hline \downarrow \quad \downarrow \\ \hline -\frac{(1+\mu)}{2} \quad \frac{(1+\mu)}{2} \end{array} \\ \hline \end{array} \right\} \bar{u} +$$

$$\left\{ \begin{array}{c} \begin{array}{c} (B_1 y_{n-1}) \\ \hline (B_4 y_n + B_5 x_{m-1}) \\ \hline \downarrow \quad \downarrow \\ \hline \otimes \\ \hline (B_4 y_n - B_5 x_{m-1}) \\ \hline (B_2 - B_3 x_m) \\ \hline (B_1 y_{n+1}) \end{array} \\ \hline \end{array} \right\} \bar{w} + \frac{L^2}{D} \bar{p}_y J_n = 0$$

Equilibrium Equation in the y-direction - J_n



Equilibrium Equation in the normal direction - JK

3.7 Strain-Displacement Relationships for Model B

The displacement system B, defined in section 3.3 has been used to develop Model B.

Consider the normal strain in the deformable element JK in the tangential direction in the x-z plane. This strain is caused by u, w displacements and the contributions to the strain by these displacements are calculated independently.

Strain in the middle surface due to u displacements

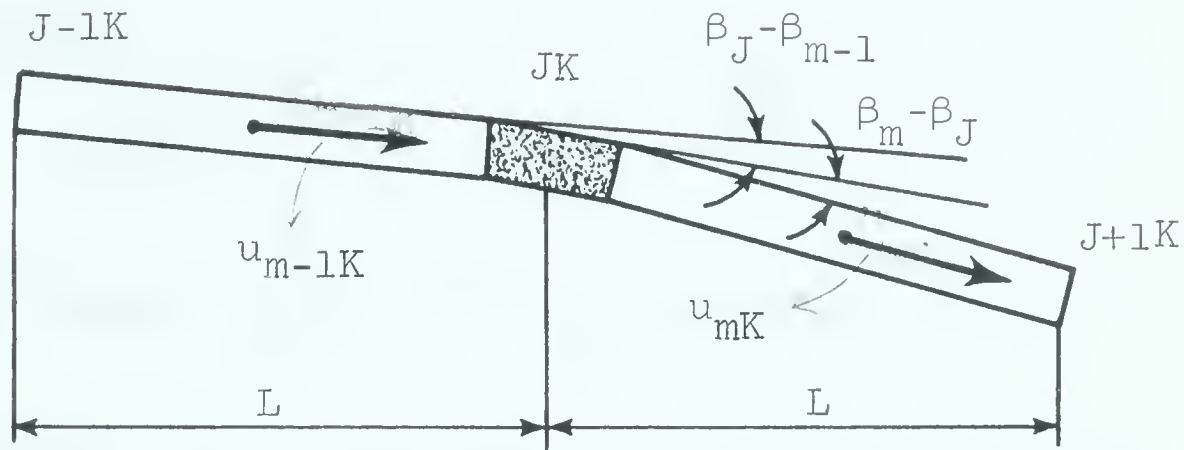
$$\begin{aligned}
 &= \frac{1}{L} \{u_{mK} \cos(\beta_m - \beta_J) - u_{m-1K} \cos(\beta_J - \beta_{m-1})\} \\
 &\approx \frac{1}{L} (u_{mK} - u_{m-1K})
 \end{aligned}$$

Strain in the middle surface due to w displacement

$$\begin{aligned}
 &= -\frac{1}{L} \{w_{JK} \tan(\beta_m - \beta_J) + w_{JK} \tan(\beta_J - \beta_{m-1})\} \\
 &\approx -\frac{w_{JK}}{L} \{\tan \beta_m - \tan \beta_{m-1}\} \\
 &= -\frac{w_{JK}}{R_1}
 \end{aligned}$$

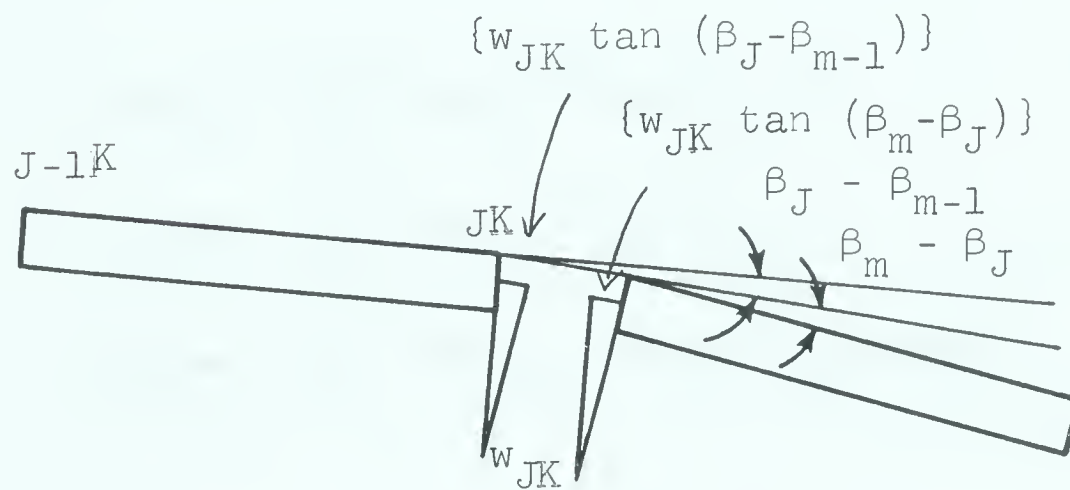
The displacements w_{J+1K} and w_{J-1K} do not produce any strain in the middle surface at the point JK.

The total normal strain ϵ_x at the point JK is given by



$$\begin{aligned}
 & \text{normal strain (tangential)} \\
 &= \frac{1}{L} \{u_{mK} \cos (\beta_m - \beta_J) - u_{m-1K} \cos (\beta_J - \beta_{m-1})\} \\
 &\approx \frac{1}{L} (u_{mK} - u_{m-1K})
 \end{aligned}$$

(a) Effect of u displacement on normal strain



$$\begin{aligned}
 \text{normal strain} &= -\frac{1}{L} \{w_{JK} \tan(\beta_J - \beta_{m-1}) + w_{JK} \tan(\beta_m - \beta_J)\} \\
 &\approx -\frac{w_{JK}}{R_1}
 \end{aligned}$$

(b) Effect of w displacement on normal strain

Fig. 3.9 EFFECT OF DISPLACEMENTS ON NORMAL STRAIN - MODEL B

$$\epsilon_x)_{JK} = \frac{1}{L} (u_{mK} - u_{m-1K}) - \frac{w_{JK}}{R_1} \quad 3.25$$

Similarly,

$$\epsilon_y)_{JK} = \frac{1}{L} (v_{Jn} - v_{Jn-1}) - \frac{w_{JK}}{R_2} \quad 3.26$$

$$\text{Shear strain, } \gamma_{xy})_{mn} = \frac{1}{L} (u_{mK+1} - u_{mK} + v_{J+1n} - v_{Jn}) \quad 3.27$$

The effect of w displacements is negligible in this case.

By making use of the assumptions made in section 3.3 regarding the w displacements, the changes in curvatures and twist can be written as

$$\chi_x)_{JK} = \frac{1}{L^2} (w_{J+1K} - 2w_{JK} + w_{J-1K}) \quad 3.28$$

$$\chi_y)_{JK} = \frac{1}{L^2} (w_{JK+1} - 2w_{JK} + w_{JK-1}) \quad 3.29$$

$$\chi_{xy})_{mn} = \frac{1}{L^2} (w_{J+1K+1} - w_{JK+1} - w_{J+1K} + w_{JK}) \quad 3.30$$

3.8 Force-Strain Relationships for Model B.

The normal strains and changes in curvatures calculated in section 3.7 are not orthogonal to each other. However, for shallow shells, they can be assumed to be orthogonal and using Hooke's Law, the following results are obtained.

$$N_{xx})_{JK} = DL \{ \epsilon_x)_{JK} + \mu \epsilon_y)_{JK} \} \quad 3.31$$

$$N_{yy})_{JK} = DL \{ \epsilon_y)_{JK} + \mu \epsilon_x)_{JK} \} \quad 3.32$$

$$N_{xy})_{mn} = DL \frac{(1-\mu)}{2} \gamma_{xy})_{mn} \quad 3.33$$

$$M_{xx})_{JK} = -KL \{ \chi_x)_{JK} + \mu \chi_y)_{JK} \} \quad 3.34$$

$$M_{yy})_{JK} = -KL \{ \chi_y)_{JK} + \mu \chi_x)_{JK} \} \quad 3.35$$

$$M_{xy})_{mn} = -KL (1-\mu) \chi_{xy})_{mn} \quad 3.36$$

3.9 Equilibrium Equations for Model B.

Three equilibrium equations are obtained by summing the forces along the primary bars and normal to the deformable element.

$$N_{xx})_{J+1K} - N_{xx})_{JK} + N_{xy})_{mn} - N_{xy})_{mn-1} + L^2 p_1)_{mK} = 0 \quad 3.37$$

$$N_{yy})_{JK+1} - N_{yy})_{JK} + N_{xy})_{mn} - N_{xy})_{m-1n} + L^2 p_2)_{Jn} = 0 \quad 3.38$$

$$\begin{aligned} V_x^+)_{JK} - V_x^-)_{JK} + V_y^+)_{JK} - V_y^-)_{JK} + N_{xx})_{JK} \{ \sin(\beta_m - \beta_J) + \sin(\beta_J - \beta_{m-1}) \} \\ + N_{yy})_{JK} \{ \sin(\alpha_n - \alpha_K) + \sin(\alpha_K - \alpha_{n-1}) \} + L^2 p_n)_{JK} = 0 \quad 3.39 \end{aligned}$$

The vertical shears, V, are obtained as in model A and are given by Eqns. 3.16 to 3.19.

Following the steps used in model A, the Eqns. 3.37 to 3.39 can be expressed in terms of the displacement components as given below.

$$\begin{aligned}
 & (u_{m+1K} - 2u_{mK} + u_{m-1K}) + \left(\frac{1-\mu}{2}\right) (u_{mK+1} - 2u_{mK} + u_{mK-1}) \\
 & + \left(\frac{1+\mu}{2}\right) (v_{J+1n} - v_{J+1n-1} - v_{Jn} + v_{Jn-1}) \\
 & - \left(\frac{1}{R_1} + \frac{\mu}{R_2}\right) (w_{J+1K} - w_{JK})L + \frac{L^2}{D} p_1)_{mK} = 0
 \end{aligned} \tag{3.40}$$

$$\begin{aligned}
 & (v_{Jn+1} - 2v_{Jn} + v_{Jn-1}) + \left(\frac{1-\mu}{2}\right) (v_{J+1n} - 2v_{Jn} + v_{J-1n}) \\
 & + \left(\frac{1+\mu}{2}\right) (u_{mK+1} - u_{mK} - u_{m-1K+1} + u_{m-1K}) \\
 & - \left(\frac{\mu}{R_1} + \frac{1}{R_2}\right) (w_{JK+1} - w_{JK})L + \frac{L^2}{D} p_2)_{Jn} = 0
 \end{aligned} \tag{3.41}$$

$$\begin{aligned}
 & \left\{ 20 + \left(\frac{1}{R_1} + \frac{2\mu}{R_1 R_2} + \frac{1}{R_2} \right) \frac{12L^4}{t^2} \right\} w_{JK}^{-8} (w_{J+1K} + w_{J-1K} + w_{JK+1} + w_{JK-1}) \\
 & + 2 (w_{J+1K+1} + w_{J+1K-1} + w_{J-1K+1} + w_{J-1K-1}) \\
 & + (w_{J+2K} + w_{J-2K} + w_{JK+2} + w_{JK-2}) \\
 & - \frac{12L^3}{t^2} \left\{ \left(\frac{1}{R_1} + \frac{\mu}{R_2} \right) (u_{mK} - u_{m-1K}) + \left(\frac{\mu}{R_1} + \frac{1}{R_2} \right) (v_{Jn} - v_{Jn-1}) \right\} \\
 & - \frac{L^4}{K} p_n)_{JK} = 0
 \end{aligned} \tag{3.42}$$

Operator form of Eqns. 3.40 to 3.42 are given below.

$$+ \left[\begin{array}{c} -D_1 \\ \otimes \\ D_1 \end{array} \right] w + \frac{L^2}{D} p_1)_{mK} = 0$$

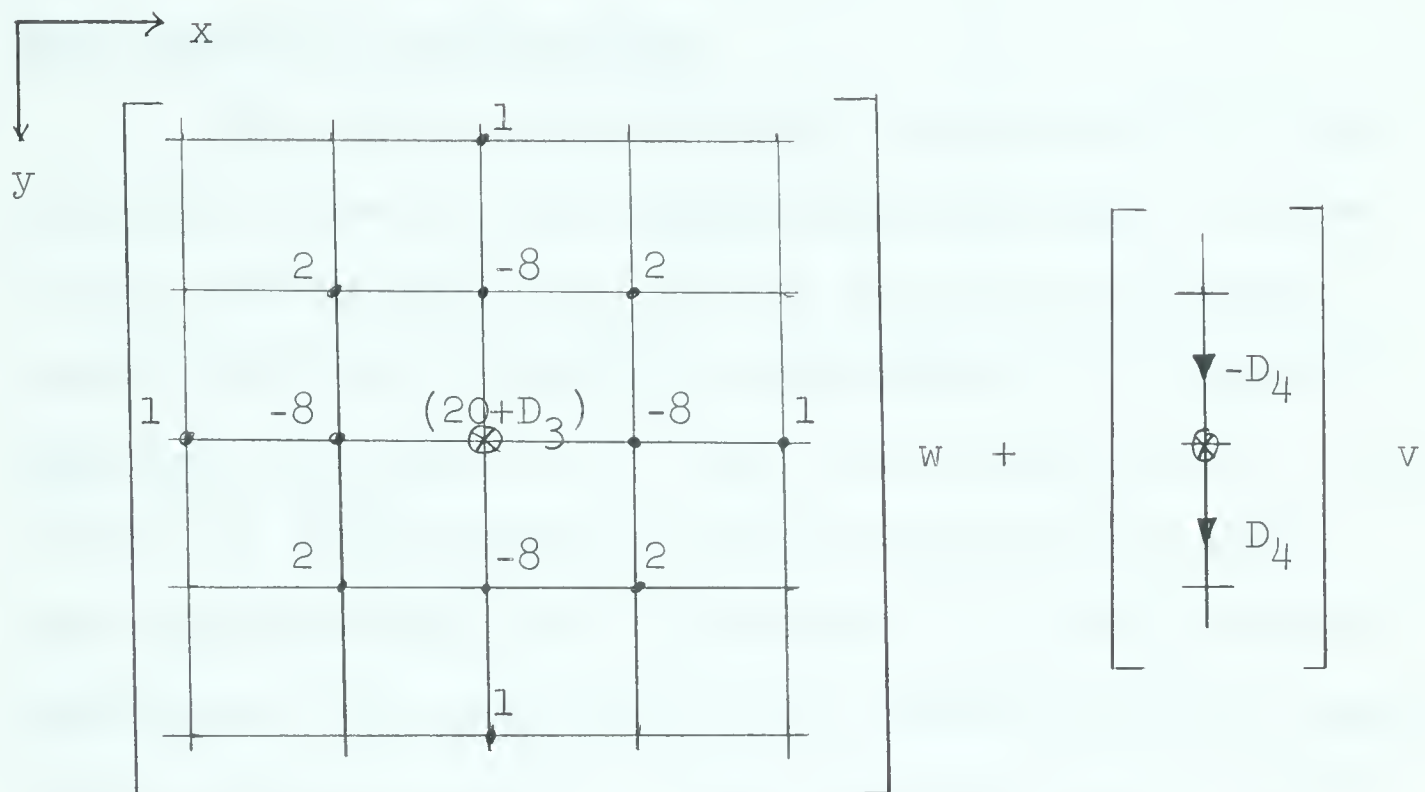
where $D_1 = -\left(\frac{1}{R_1} + \frac{\mu}{R_2}\right)L$

Equilibrium Equation in the v-direction - Point mK

$$+ \left[\begin{array}{c} -D_2 \\ \otimes \\ D_2 \end{array} \right] w + \frac{L^2}{D} p_2)_{Jn} = 0$$

where $D_2 = -\left(\frac{\mu}{R_1} + \frac{1}{R_2}\right)L$

Equilibrium Equation in the u-direction - Point Jn



$$+ \left[\begin{array}{c} -D_5 \\ \rightarrow \\ \text{---} \end{array} \right] u - \frac{L^4}{K} p_n)_{JK} = 0$$

where $D_3 = \left\{ \frac{1}{R_1^2} + \frac{2\mu}{R_1 R_2} + \frac{1}{R_2^2} \right\} \frac{12L^4}{t^2}$

$$D_4 = \frac{-12L^3}{t^2} \left(\frac{1}{R_1} + \frac{\mu}{R_2} \right)$$

$$D_5 = \frac{-12L^3}{t^2} \left(\frac{\mu}{R_1} + \frac{1}{R_2} \right)$$

Equilibrium Equation in the w-direction - Point JK

3.10 Method of application

The shell is divided into a square grid on the projected plane and the equilibrium equations in terms of the displacements are written for every deformable element and every primary bar using Model A or Model B. Equations for points on or near the boundary will involve fictitious displacements of points outside the shell. These displacements can be related to the real displacements using the conditions at the boundaries (see Chapter V). Using a digital computer, these equations can be solved to obtain the unknown displacements. Knowing the displacements, the forces in the shell can be calculated using the strain-displacement relationships and the force-strain relationships.

CHAPTER IV

COMPARISON OF THE MODELS WITH THE CLASSICAL THEORIES

4.1 Introduction.

Two sets of equilibrium equations were obtained in Chapter III from the two physical models and it is necessary to check the validity of these equations against the classical theories. The displacement system of model A is similar to that used by Marguerre and the displacement system of model B is similar to that used by Conrad. Hence, the equations of model A should be compared with Marguerre's theory and the equations of model B should be compared with Conrad's theory.

Since the equations obtained from the models are similar in form to finite difference equations, the comparison is made by comparing the equations of the models with the equations obtained by applying central finite difference operators to the differential equations of the classical theories.

In a recent study of cylindrical shells [3], it had been shown that the conventional method of writing the central finite difference operators for the differential

equations of a shell with u-v-w formulation is unsatisfactory. However, in that study, a satisfactory method was presented where the u,v displacements are defined midway between the w displacements and the odd derivatives are defined at points midway between the points where the even derivatives are defined. This modified method of finite difference operators are used in this study.

The following are the modified finite difference operators.

$$f_{,X} = \frac{1}{L} \left[\begin{array}{c|ccc} & -1 & & 1 \\ \hline & X & \bullet & X \\ & | & | & | \\ & \downarrow & & \downarrow \\ & Y & & \end{array} \right] f$$

$$f_{,xx} = \frac{1}{L^2} \left[\overset{1}{\bullet} \overset{2}{\otimes} \overset{1}{\bullet} \right] f$$

$$f_{xxx} = \frac{1}{13} \left[\begin{array}{ccccccc} | & -1 & | & 3 & | & -3 & | & 1 & | \\ \hline \end{array} \right] f$$

$$f_{,xxxx} = \frac{1}{L^4} \left[\begin{array}{ccccc} 1 & -4 & 6 & -4 & 1 \end{array} \right] f$$

$$f_{,y} = \frac{1}{L} \left[\begin{array}{c} \text{---} \\ | \\ \bullet \quad -1 \\ | \\ \otimes \rightarrow x \\ | \\ \bullet \quad 1 \\ | \\ \text{---} \\ \downarrow y \end{array} \right] \quad f; \quad f_{,yy} = \frac{1}{L^2} \left[\begin{array}{c} \text{---} \\ | \\ 1 \\ | \\ \otimes -2 \\ | \\ 1 \\ | \\ \text{---} \end{array} \right] f;$$

$$f_{,yyy} = \frac{1}{L^3} \left[\begin{array}{c} \text{---} \\ | \\ \bullet \quad 1 \\ | \\ \text{---} \\ | \\ \bullet \quad 3 \\ | \\ \otimes \\ | \\ \bullet \quad -3 \\ | \\ \text{---} \\ | \\ \bullet \quad 1 \\ | \\ \text{---} \end{array} \right] \quad f; \quad f_{,yyy} = \frac{1}{L^4} \left[\begin{array}{c} \text{---} \\ | \\ 1 \\ | \\ \text{---} \\ | \\ -4 \\ | \\ \otimes 6 \\ | \\ -4 \\ | \\ \text{---} \\ | \\ 1 \\ | \\ \text{---} \end{array} \right] f$$

$$f_{,xy} = \frac{1}{L^2} \left[\begin{array}{ccc} & & \\ \bullet \quad 1 & & \bullet \quad -1 \\ & \otimes & \\ \bullet \quad -1 & & \bullet \quad 1 \\ & & \end{array} \begin{array}{c} \text{---} \\ | \\ \rightarrow x \\ | \\ \downarrow y \end{array} \right] f$$

$$f_{,xxyy} = \frac{1}{L^4} \left[\begin{array}{ccc} & & \\ 1 & -2 & 1 \\ & \otimes 4 & \\ -2 & & -2 \\ & & \\ 1 & -2 & 1 \\ & & \end{array} \begin{array}{c} \text{---} \\ | \\ \rightarrow x \\ | \\ \downarrow y \end{array} \right] f$$

where the function f can be interpreted as any one of the displacement components and L is the length of the grid.

It should be noted that the displacement systems used in the models are in agreement with this concept of finite difference operators.

4.2 Finite Difference Equations for Marguerre's Equations.

In the equations obtained by Marguerre's theory, there are two variable functions, one due to the geometry of the shell and the other due to the displacements. In the final form of the equations, these functions appear as a product. However, for the application of the finite difference operators, these functions are defined at different positions on the shell surface and so operators are applied to the expressions prior to the final differentiation.

Consider the equilibrium equation (see Appendix A, Eqn. A.17)

$$N_{xx,x} + N_{xy,y} + \bar{p}_x = 0 \quad 4.1$$

which can be written in terms of the strains as

$$(\bar{\epsilon}_x + \mu \bar{\epsilon}_y)_{,x} + \left\{ \frac{(1-\mu)}{2} \gamma_{xy} \right\}_{,y} + \frac{p_x}{D} = 0 \quad 4.2$$

and in terms of the displacements as

$$\begin{aligned} & (\bar{u}_{,x})_{,x} + (\mu \bar{v}_{,y})_{,x} + \frac{(1-\mu)}{2} (\bar{u}_{,y})_{,y} + \frac{(1-\mu)}{2} (\bar{v}_{,x})_{,y} \\ & + (z_{,x} \bar{w}_{,x})_{,x} + \mu (z_{,y} \bar{w}_{,y})_{,x} + \frac{(1-\mu)}{2} (z_{,x} \bar{w}_{,y})_{,y} \\ & + \frac{(1-\mu)}{2} (z_{,y} \bar{w}_{,x})_{,y} + \frac{\bar{p}_x}{D} = 0 \end{aligned} \quad 4.3$$

The finite difference equation for Eqn. 4.3 for the point mK can be written as

$$\begin{aligned} & \frac{1}{L^2} (\bar{u}_{m+1K} - 2\bar{u}_{mK} + \bar{u}_{m-1K}) + \frac{(1+\mu)}{2L^2} (\bar{v}_{J+1n} - \bar{v}_{J+1n-1} - \bar{v}_{Jn} + \bar{v}_{Jn-1}) \\ & + \frac{1}{L^2} \frac{(1-\mu)}{2} (\bar{u}_{mK+1} - 2\bar{u}_{mK} + \bar{u}_{mK-1}) \\ & + \frac{1}{2L} \left[\frac{1}{L} \left\{ (\bar{w}_{J+2K} - \bar{w}_{J+1K}) (z_{,x})_{m+1} - (\bar{w}_{JK} - \bar{w}_{J-1K}) (z_{,x})_{m-1} \right\} \right] \\ & + \frac{\mu}{L} \left[\frac{1}{2L} \left\{ (\bar{w}_{J+1K+1} - \bar{w}_{J+1K}) (z_{,y})_n + (\bar{w}_{J+1K} - \bar{w}_{J+1K-1}) (z_{,y})_{n-1} \right\} \right. \\ & \left. - \frac{1}{2L} \left\{ (\bar{w}_{JK+1} - \bar{w}_{JK}) (z_{,y})_n + (\bar{w}_{JK} - \bar{w}_{JK-1}) (z_{,y})_{n-1} \right\} \right] \\ & + \frac{(1-\mu)}{2L} \left[\frac{1}{2L} \left\{ (\bar{w}_{J+1K+1} - \bar{w}_{J+1K} + \bar{w}_{JK+1} - \bar{w}_{JK}) (z_{,x})_m - (\bar{w}_{J+1K} - \bar{w}_{J+1K-1} \right. \right. \\ & \left. \left. + \bar{w}_{JK} - \bar{w}_{JK-1}) (z_{,x})_{m-1} \right\} \right] \\ & + \frac{(1-\mu)}{2L} \left[\frac{1}{2L} \left\{ (\bar{w}_{J+1K+1} + \bar{w}_{J+1K} - \bar{w}_{JK+1} - \bar{w}_{JK}) (z_{,y})_n - (\bar{w}_{J+1K} + \bar{w}_{J+1K-1} - \right. \right. \\ & \left. \left. \bar{w}_{JK} - \bar{w}_{JK-1}) (z_{,y})_{n-1} \right\} \right] + \frac{\bar{p}_x}{D} = 0 \end{aligned} \quad 4.4$$

Eqn. 4.4 can be simplified by substituting for

$$z = \frac{x^2}{2R_1} + \frac{y^2}{2R_2}$$

leaving,

$$\begin{aligned} & (\bar{u}_{m+1K} - 2\bar{u}_{mK} + \bar{u}_{m-1K}) + \frac{(1-\mu)}{2} (\bar{u}_{mK+1} - 2\bar{u}_{mK} + \bar{u}_{mK-1}) \\ & + \frac{(1+\mu)}{2} (\bar{v}_{J+1n} - \bar{v}_{J+1n-1} - \bar{v}_{Jn} + \bar{v}_{Jn-1}) \\ & + (A_1 x_{m+1}) \bar{w}_{J+2K} + (A_2 - A_3 x_m) \bar{w}_{J+1K} \\ & - (A_2 + A_3 x_m) \bar{w}_{JK} + (A_1 x_{m-1}) \bar{w}_{J-1K} \\ & + (A_4 x_m + A_5 y_n) \bar{w}_{J+1K+1} + (A_4 x_m - A_5 y_n) \bar{w}_{JK+1} \\ & + (A_4 x_m - A_5 y_{n-1}) \bar{w}_{J+1K-1} + (A_4 x_m + A_5 y_{n-1}) \bar{w}_{JK-1} \\ & + \frac{\bar{p}_x L^2}{D} = 0 \end{aligned} \tag{4.5}$$

where A_1 to A_5 are defined in section 3.6.

Similarly, by considering the equilibrium equation in the y-direction, the following equation is obtained.

$$\begin{aligned}
& (\bar{v}_{Jn+1} - 2\bar{v}_{Jn} + \bar{v}_{Jn-1}) + \frac{(1-\mu)}{2} (\bar{v}_{J+1n} - 2\bar{v}_{Jn} + \bar{v}_{J-1n}) \\
& + \frac{(1+\mu)}{2} (\bar{u}_{mK+1} - \bar{u}_{mK} - \bar{u}_{m-1K+1} + \bar{u}_{m-1K}) + \\
& (B_1 y_{n+1}) \bar{w}_{JK+2} + (B_2 - B_3 y_n) \bar{w}_{JK+1} - (B_2 + B_3 y_n) \bar{w}_{JK} + (B_1 y_{n-1}) \bar{w}_{JK-1} \\
& + (B_4 y_n + B_5 x_m) \bar{w}_{J+1K+1} + (B_4 y_n - B_5 x_m) \bar{w}_{J-1K} \\
& + (B_4 y_n - B_5 x_{m-1}) \bar{w}_{J-1K+1} + (B_4 y_n + B_5 x_{m-1}) \bar{w}_{J-1K} \\
& + \bar{p}_y \frac{L^2}{D} = 0
\end{aligned} \tag{4.6}$$

where B_1 to B_5 are defined in section 3.6.

Now, consider the equilibrium equation in the normal direction as given below. (see appendix A. Eqn. A.27)

$$\begin{aligned}
& \left(\frac{1}{R_1} + \frac{\mu}{R_2}\right) \bar{u}_{,x} + \left(\frac{\mu}{R_1} + \frac{1}{R_2}\right) \bar{v}_{,y} + \left(\frac{1}{R_1} + \frac{\mu}{R_2}\right) z_{,x} \bar{w}_{,x} \\
& + \left(\frac{\mu}{R_1} + \frac{1}{R_2}\right) z_{,y} \bar{w}_{,y} - \frac{t^2}{12} \nabla^2 \nabla^2 \bar{w} + \frac{p_n}{D} = 0
\end{aligned} \tag{4.7}$$

Finite difference equation for Eqn. 4.7 for point JK can be written as

$$\begin{aligned}
& \frac{1}{L} \left(\frac{1}{R_1} + \frac{\mu}{R_2} \right) (\bar{u}_{mK} - \bar{u}_{m-1K}) + \frac{1}{L} \left(\frac{\mu}{R_1} + \frac{1}{R_2} \right) (\bar{v}_{Jn} - \bar{v}_{Jn-1}) + \\
& \frac{1}{2L} \left(\frac{1}{R_1} + \frac{\mu}{R_2} \right) \left\{ (z_{,x})_m (\bar{w}_{J+1K} - \bar{w}_{JK}) + (z_{,x})_{m-1} (\bar{w}_{JK} - \bar{w}_{J-1K}) \right\} + \\
& \frac{1}{2L} \left(\frac{\mu}{R_1} + \frac{1}{R_2} \right) \left\{ (z_{,y})_n (\bar{w}_{JK+1} - \bar{w}_{JK}) + (z_{,y})_{n-1} (\bar{w}_{JK} - \bar{w}_{JK-1}) \right\} \\
& - \frac{t^2}{12L^4} \left\{ 20w_{JK} - 8(\bar{w}_{J+1K} + \bar{w}_{J-1K} + \bar{w}_{JK+1} + \bar{w}_{JK-1}) \right. \\
& \quad + 2(\bar{w}_{J+1K+1} + \bar{w}_{J+1K-1} + \bar{w}_{J-1K+1} + \bar{w}_{J-1K-1}) \\
& \quad \left. + (w_{J+2K} + w_{J-2K} + w_{JK+2} + w_{JK-2}) \right\} \\
& + \frac{p_n}{D} = 0 \tag{4.8}
\end{aligned}$$

Eqn. 4.8 can be simplified as follows.

$$\begin{aligned}
& (20+C_1)\bar{w}_{JK} - (8+C_2x_m)\bar{w}_{J+1K} - (8-C_2x_{m-1})\bar{w}_{J-1K} \\
& - (8+C_2y_n)w_{JK+1} - (8-C_2y_{n-1})w_{JK-1} \\
& + 2(\bar{w}_{J+1K+1} + \bar{w}_{J+1K-1} + \bar{w}_{J-1K+1} + \bar{w}_{J-1K-1}) \\
& + (\bar{w}_{J+2K} + \bar{w}_{J-2K} + \bar{w}_{JK+2} + \bar{w}_{JK-2}) \\
& - C_4(\bar{u}_{mK} - \bar{u}_{m-1K}) - C_5(\bar{v}_{Jn} - \bar{v}_{Jn-1}) - \frac{p_n L^4}{K} = 0 \tag{4.9}
\end{aligned}$$

where C_1 to C_5 are defined in section 3.6.

Eqns. 4.4, 4.5 and 4.9 are identically the same as

Eqns. 3.22 to 3.24 obtained from model A.

4.3 Finite Difference Equations for Conrad's Equations.

Since the governing equations by Conrad's theory are independent of the position on the shell surface, the finite difference equations can be written more readily. Consider Eqn. 4.11 (see appendix B Eqn. B.35)

$$u_{,xx} + \frac{(1-\mu)}{2}u_{,yy} + \frac{(1+\mu)}{2}v_{,xy} - \left(\frac{1}{R_1} + \frac{\mu}{R_2}\right)w_{,x} + \frac{p_1}{D} = 0 \quad 4.10$$

The finite difference equation for the point mK can be written as

$$\begin{aligned} & (u_{m+1K} - 2u_{mK} + u_{m-1K}) + \frac{(1-\mu)}{2}(u_{mK+1} - 2u_{mK} + u_{mK-1}) \\ & + \frac{(1+\mu)}{2}(v_{J+1n} - v_{J+1n-1} - v_{Jn} + v_{Jn-1}) \\ & - L\left(\frac{1}{R_1} + \frac{\mu}{R_2}\right)(w_{J+1K} - w_{JK}) + \frac{p_1 L^2}{D} = 0 \end{aligned} \quad 4.11$$

Similarly, by considering the other two equilibrium equations, the finite difference equations for the points Jn and JK can be written as

$$\begin{aligned} & (v_{Jn+1} - 2v_{Jn} + v_{Jn-1}) + \frac{(1-\mu)}{2}(v_{J+1n} - 2v_{Jn} + v_{J-1n}) \\ & + \frac{(1+\mu)}{2}(u_{mK+1} - u_{mK} - u_{m-1K+1} + u_{m-1K}) \\ & - L\left(\frac{\mu}{R_1} + \frac{1}{R_2}\right)(w_{JK+1} - w_{JK}) + \frac{p_2 L^2}{D} = 0 \end{aligned} \quad 4.12$$

$$\begin{aligned}
& \left\{ 20 + \left(\frac{1}{R_1} + \frac{2\mu}{R_1 R_2} + \frac{1}{R_2} \right) \frac{12L^4}{t^2} \right\} w_{JK} - 8(w_{J+1K} + w_{J-1K} + w_{JK+1} + w_{JK-1}) \\
& + 2(w_{J+1K+1} + w_{J-1K+1} + w_{J+1K-1} + w_{J-1K-1}) \\
& + 1(w_{J+2K} + w_{J-2K} + w_{JK+2} + w_{JK-2}) \\
& - \frac{12L^3}{t^2} \left\{ \left(\frac{1}{R_1} + \frac{\mu}{R_2} \right) (u_{mK} - u_{m-1K}) + \left(\frac{\mu}{R_1} + \frac{1}{R_2} \right) (v_{Jn} - v_{Jn-1}) \right\} \\
& - \frac{L^4 p_n}{K} = 0
\end{aligned} \tag{4.13}$$

Eqns. 4.11 to 4.13 are identically the same as Eqns. 3.40 to 3.42.

4.4 Conclusions.

It has been shown in this chapter that the equations obtained from the models correspond to the finite difference equations obtained from the classical theories. Hence, it can be assumed that the principles used in developing the models are valid and the models can be used for analysing the shell structures.

CHAPTER V

BOUNDARY CONDITIONS

5.1 Introduction.

The method of analysis using the physical models is similar to the method of finite differences. The shell is first divided into a square grid and the equilibrium equations are written at every intersection of the grid lines and at the mid-point of the grid lines. This will result in a set of linear, algebraic, simultaneous equations in terms of the unknown displacements. However, it will be found that the number of equations obtained in this way is insufficient to solve for the unknown displacements due to the fact that the equations written for points on or near the boundaries involve displacements of fictitious points outside the shell. Therefore it is necessary to write additional equations or to modify the models at the boundaries to eliminate these unknown displacements of the fictitious points. In this way, sufficient equations can be obtained to solve for the unknown displacements.

It is anticipated that Model B will be used more often than Model A since the equations of Model B are much simpler. Hence, the additional equations and modifications necessary only for Model B for some types of boundaries are discussed in this chapter.

The types of boundaries considered are

(i) shells with simply supported edges: Strictly speaking this type of boundary very seldom exists in practice. However, shells supported on beams that are rigid in their plane but flexible in planes normal to them can be assumed to be 'simply supported' provided the torsional rigidities of the beams are small.

(ii) Shells with fixed edges: A boundary that prevents both displacements and rotations of the shell edge is classified as 'fixed edge.'

(iii) Shells with free edges: Any edge that is not restrained against displacements and rotation is called a 'free edge'.

(iv) Shells supported by elastic beams: This type of boundary is the most generally encountered one in practice. Here, the edge member, which is placed to carry the edge shear, permits both finite displacements and rotation of the shell edge.

Cases (i) to (iii) are special cases of (iv).

5.2 Shells with simply supported edges.

For this type of boundary, four independent conditions at the edge are used to obtain the additional equation necessary to solve for the unknowns.

At edge $x=J$, these conditions are

$$\left. \begin{aligned} N_{xx} &= 0 \\ M_{xx} &= 0 \\ v &= 0 \\ w &= 0 \end{aligned} \right\} \quad 5.1$$

These conditions can be expressed in terms of the displacements as follows.

$$\left. \begin{aligned} w_{JK} &= 0 \\ v_{Jn} &= 0 \\ w_{J+1 \ K} &= -w_{J-1 \ K} \\ u_{mK} &= u_{m-1 \ K} \end{aligned} \right\} \quad 5.2$$

From Eqn. 5.2, it can be seen that it is not necessary to write the equilibrium equations at the points JK and Jn on the edge since the displacements are known to be zero.

5.3 Shells with fixed edges.

Again, four additional boundary conditions are defined to provide sufficient equations to solve for the unknowns. At edge $x = J$, these conditions are

$$\left. \begin{aligned} v &= 0 \\ w &= 0 \\ u &= 0 \\ \text{change of slope} &= 0 \end{aligned} \right\} \quad 5.3$$

The above conditions can be expressed in terms of the displacement components of the model as

$$\left. \begin{aligned} v_{Jn} &= 0 \\ w_{JK} &= 0 \\ u_{mK} &= -u_{m-1 K} \\ w_{J+1 K} &= w_{J-1 K} \end{aligned} \right\} \quad 5.4$$

The equilibrium equations for the points JK and Jn can be avoided since the displacements at these points are already known.

5.4 Shells with free edges.

It is found easier to modify the equilibrium equations for points on or near this type of boundary instead of writing additional equations. At edge $x = J$, the following conditions exist.

$$\left. \begin{aligned} M_{xx} &= 0 \\ N_{xx} &= 0 \\ V_x^+)_{JK} &= 0 \\ N_{xy}^-)_{mn} &= 0 \end{aligned} \right\} \quad 5.5$$

From the first two conditions of Eqn. 5.5, the following results are obtained

$$\chi_x)_{JK} = -\mu \chi_y)_{JK}$$

$$\epsilon_x)_{JK} = -\mu \epsilon_y)_{JK}$$

Hence,

$$M_{yy})_{JK} = -\frac{K}{2} \{1 - \mu^2\} \chi_y)_{JK} \quad 5.6$$

$$N_{yy})_{JK} = +\frac{D}{2} \{1 - \mu^2\} \epsilon_y)_{JK} \quad 5.7$$

Note that the stiffness of the shell is due to half the grid length of the shell only in Eqns. 5.6 and 5.7.

The modified equilibrium equations for the points JK, Jn, m-1 K and J-1 K are given below.

Point JK

$$\begin{aligned}
& \frac{1}{2}(1-\mu^2) W_{JK+2} + (-4 + 2\mu + 2\mu^2) W_{JK+1} + \{8 - 4\mu - 3\mu^2 + \\
& \frac{(1-\mu^2)12L^4}{2t^2 R_2^2} \} W_{JK} + (-4 + 2\mu + 2\mu^2) W_{JK-1} \\
& + \frac{1}{2}(1-\mu^2) W_{JK-2} + (2 - \mu) W_{J-1 K+1} \\
& + (-6 + 2\mu) W_{J-1 K} + (2 - \mu) W_{J-1 K-1} + W_{J-2K} \\
& - \frac{12L^3}{2t^2 R_2} (1-\mu^2) (v_{Jn} - v_{Jn-1}) - \frac{p_n L^4}{2K} = 0 \quad 5.8
\end{aligned}$$

Point Jn

$$\begin{aligned}
& \left(\frac{1-\mu^2}{2}\right) v_{Jn+1} + \left[- \{ (1-\mu^2) + \left(\frac{1-\mu}{2}\right) \} \right] v_{Jn} + \frac{1-\mu^2}{2} v_{Jn-1} \\
& \left(\frac{1-\mu}{2}\right) v_{J-1 n} - \left(\frac{1-\mu}{2}\right) u_{m-1 K+1} + \left(\frac{1-\mu}{2}\right) u_{m-1 K} \\
& - \frac{(1-\mu^2)L}{2R_2} (w_{JK+1} - w_{JK}) + \frac{p_2 L^2}{2D} = 0 \quad 5.9
\end{aligned}$$

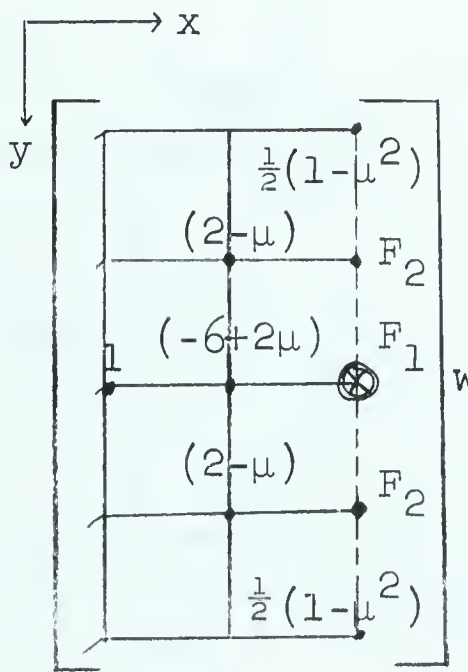
Point m-1 K

$$\begin{aligned}
& (-2+\mu) u_{m-1 K} + u_{m-2 K} + \left(\frac{1-\mu}{2}\right) u_{m-1 K+1} + \left(\frac{1-\mu}{2}\right) u_{m-1 K-1} \\
& + \left(\frac{1-\mu}{2}\right) v_{Jn} - \left(\frac{1-\mu}{2}\right) v_{Jn-1} - \left(\frac{1+\mu}{2}\right) v_{J-1 n} + \left(\frac{1+\mu}{2}\right) v_{J-1 n-1} \\
& + L \left(\frac{1}{R_1} + \frac{\mu}{R_2}\right) w_{J-1 K} + \frac{p_1 L^2}{D} = 0 \quad 5.10
\end{aligned}$$

Point J-1 K

$$\begin{aligned}
 & (2-\mu) w_{JK+1} + (-6 + 2\mu) w_{JK} + (2-\mu) w_{JK-1} - w_{J-1 K+2} \\
 & -8 w_{J-1 K+1} + \left\{ 19 + \frac{12L^4}{t^2} \left(\frac{1}{R_1^2} + \frac{2\mu}{R_1 R_2} + \frac{1}{R_2^2} \right) \right\} w_{JK} \\
 & -8 w_{J-1 K-2} + 2 w_{J-2 K+1} - 8 w_{J-2 K} + 2 w_{J-2 K-1} \\
 & + w_{J-3 K} - \frac{12L^3}{t^2} \left(\frac{1}{R_1} + \frac{\mu}{R_2} \right) (u_{m-1 K} - u_{m-2 K}) \\
 & - \frac{12L^3}{t^2} \left(\frac{\mu}{R_1} + \frac{1}{R_2} \right) (v_{Jn} - v_{Jn-1}) - \frac{p_n L^4}{K} = 0 \quad 5.11
 \end{aligned}$$

Equations 5.8 to 5.11 are independent of the deflections of the fictitious points and hence there is no need for additional equations. The operator forms of these equations are given below.



$$\begin{aligned}
 & \left[\begin{array}{c} \downarrow F_3 \\ \otimes F_3 \\ \downarrow F_3 \end{array} \right] v - \frac{p_n L^4}{2K} = 0 \\
 & F_1 = \left\{ 8 - 4\mu - 3\mu^2 + \frac{(1-\mu^2)12L^4}{t^2 R_2^2} \right\} \\
 & F_2 = (4\mu^2 + 2\mu - 4) \\
 & F_3 = - \frac{12L^3}{t^2 R_2} (1-\mu^2)
 \end{aligned}$$

Equilibrium equation in the w-direction--Point JK

$$F_7 = - \left\{ (1-\mu^2) + \frac{(1-\mu)}{2} \right\}$$

Equilibrium Equation in the v-direction--Point Jn

$$w + \frac{p_1 L^2}{D} = 0$$

Equilibrium Equation in the u-direction--Point m-1 K

$$\left[\begin{array}{ccc} & 2 & -8 & (2-\mu) \\ 1 & -8 & F_4 & -(6-2\mu) \\ & 2 & -8 & (2-\mu) \end{array} \right] w + \left[\begin{array}{c} \text{Diagram of forces } -F_5 \text{ and } F_5 \end{array} \right] u +$$

$$+ \left[\begin{array}{c} \text{Diagram of forces } -F_6 \text{ and } F_6 \end{array} \right] v - \frac{p_n L^4}{K} = 0$$

$$F_4 = \left\{ 19 + \frac{12L^4}{t^2} \left(\frac{1}{R_1^2} + \frac{2\mu}{R_1 R_2} + \frac{1}{R_2^2} \right) \right\}$$

$$F_5 = - \frac{12L^3}{t^2} \left(\frac{1}{R_1} + \frac{\mu}{R_2} \right)$$

$$F_6 = - \frac{12L^3}{t^2} \left(\frac{\mu}{R_1} + \frac{1}{R_2} \right)$$

Equilibrium Equation in the w-direction--Point J-1 K

5.5 Shells supported by elastic beams.

It is found necessary to modify the model at the edges for this type of boundary as shown in Fig. 3.5. The forces in the shell and beam are considered separately.

Due to the modifications in the model, it is necessary to define a new displacement component and then recalculate the strains and forces. The tangential displacement component of the beam in a plane normal to that of the beam is defined at the deformable element in the beam.

At edge $x=J$, the axial forces in the shell are

$$N_{xx})_{JK} = DL \left[\left\{ \frac{2(u_{JK} - u_{m-1 K})}{L} - \frac{w_{JK}}{R_1} \right\} + \mu \left\{ \frac{(v_{Jn} - v_{Jn-1})}{L} - \frac{w_{JK}}{R_2} \right\} \right] \quad 5.12$$

$$N_{yy})_{JK} = \frac{DL}{2} \left[\left\{ \frac{(v_{Jn} - v_{Jn-1})}{L} - \frac{w_{JK}}{R_2} \right\} + \mu \left\{ \frac{2(u_{JK} - u_{m-1 K})}{L} - \frac{w_{JK}}{R_1} \right\} \right] \quad 5.13$$

Note that the stiffness of the shell in the y-direction is due to half a grid length of the shell only.

The axial force in the beam is

$$N'_{yy})_{JK} = DL \lambda_1 \left\{ \frac{v_{Jn} - v_{Jn-1}}{L} - \frac{w_{JK}}{R_2} \right\} \quad 5.14$$

where λ_1 = ratio of the axial stiffness of the beam to that of the shell of width L.

The bending moments in the beam in the plane of the beam and in the plane of the shell are given by

$$M'_{yy})_{JK} = - \frac{KL\lambda_2}{L^2} (w_{JK+1} - 2w_{JK} + w_{JK-1}) \quad 5.15$$

$$M''_{yy})_{JK} = - \frac{KL\lambda_4}{L^2} (u_{JK+1} - 2u_{JK} + u_{JK-1}) \quad 5.16$$

where λ_2 = ratio of the bending stiffness of the beam in a plane normal to the shell to that of the shell of width L.

λ_4 = ratio of the bending stiffness of the beam in the plane of the shell to that of the shell of width L.

The torsional moment in the beam is given by

$$T_{Jn} = -KL\lambda_3 \left\{ \frac{1}{2L^2} (w_{J+1 \ K+1} - w_{J-1 \ K+1} - w_{J+1 \ K} + w_{J-1 \ K}) \right\} \quad 5.17$$

where λ_3 = ratio of the torsional rigidity of the beam to bending stiffness of the shell of width L .

By summing the forces along the bars JK+1-JK and JK - J-1 K and normal to the deformable element JK, the following equilibrium equations are obtained.

$$N_{yy})_{JK+1} + N'_{yy})_{JK+1} - N_{yy})_{JK} - N'_{yy})_{JK} - N_{xy})_{m-1n} + \frac{p_2 L^2}{2} = 0 \quad 5.18$$

$$N_{xx})_{JK} - N_{xx})_{J-1 K} + N_{xy})_{m-1n} - N_{xy})_{m-1 n-1} + p_1 L^2 = 0 \quad 5.19$$

$$\begin{aligned} & \left\{ V_y^+)_{JK} + V_y^{+'})_{JK} \right\} - \left\{ V_y^-)_{JK} + V_y^{-'})_{JK} \right\} - V_x^-)_{JK} \\ & + \left\{ N_{yy})_{JK} + N'_{yy})_{JK} \right\} \left\{ \sin (\alpha_n - \alpha_K) + \sin (\alpha_K - \alpha_{n-1}) \right\} \\ & + N_{xx})_{JK} \sin (\beta_J - \beta_{m-1}) + \frac{p_n L^2}{2} = 0 \quad 5.20 \end{aligned}$$

The vertical shears in Eqn. 5.20 can be expressed in terms of the moments by considering the moment equilibrium of the bars as follows.

$$\left\{ V_y^+ \right\}_{JK} + \left\{ V_y^{+'} \right\}_{JK} \right\} = \frac{1}{L} \left\{ M_{yy} \right\}_{JK+1} + M'_{yy} \right\}_{JK+1} - M_{yy} \right\}_{JK} - M'_{yy} \right\}_{JK} - M_{xy} \right\}_{m-1n} \right\} \quad 5.21$$

$$\left\{ V_y^- \right\}_{JK} + \left\{ V_y^{-'} \right\}_{JK} \right\} = \frac{1}{L} \left\{ M_{yy} \right\}_{JK} + M'_{yy} \right\}_{JK} - M_{yy} \right\}_{JK-1} - M'_{yy} \right\}_{JK-1} - M_{xy} \right\}_{m-1 \ n-1} \right\} \quad 5.22$$

$$V_x^- \right\}_{JK} = \frac{1}{L} \left\{ M_{xx} \right\}_{JK} - M_{xx} \right\}_{J-1 \ K} + M_{xy} \right\}_{m-1n} - M_{xy} \right\}_{m-1 \ n-1} \right\} \quad 5.23$$

Equilibrium equations can now be expressed in terms of the displacements by substituting for the stress resultants obtained above in terms of the displacements. Thus, the Eqns. 5.18 to 5.20 become

$$\begin{aligned} & \left(\frac{1}{2} + \lambda_1 \right) v_{Jn+1} + \left(-\frac{3}{2} - 2\lambda_1 + \frac{\mu}{2} \right) v_{Jn} + \left(\frac{1}{2} + \lambda_1 \right) v_{Jn-1} + \left(\frac{1-\mu}{2} \right) v_{J-1n} \\ & + \mu u_{JK+1} - \mu u_{JK} - \left(\frac{1+\mu}{2} \right) u_{m-1 \ K+1} + \left(\frac{1+\mu}{2} \right) u_{m-1 \ K} \\ & - \left(\frac{\mu L}{2R_1} + \frac{L}{2R_2} + \frac{L\lambda_1}{R_2} \right) (w_{JK+1} - w_{JK}) + \frac{p_2 L^2}{2D} = 0 \end{aligned} \quad 5.24$$

$$\begin{aligned}
& (2u_{JK} - 3 u_{m-1 K} + u_{m-2 K}) + \frac{(1+\mu)}{2} (v_{Jn} - v_{Jn-1} - v_{J-1n} + v_{J-1n-1}) \\
& + \frac{(1-\mu)}{2} (u_{m-1 K+1} - 2 u_{m-1 K} + u_{m-1 K-1}) \\
& - \left(\frac{L}{R_1} + \frac{L\mu}{R_2} \right) (w_{JK} - w_{J-1 K}) + \frac{p_1 L^2}{D} = 0 \quad 5.25
\end{aligned}$$

$$\begin{aligned}
& \frac{\mu}{2} w_{J+1 K+1} - (1+\mu) w_{J+1 K} + \frac{\mu}{2} w_{J+1 K-1} + \\
& (\lambda_2 + \frac{1}{2}) w_{JK+2} - 4(1+\lambda_2) w_{JK+1} + \left[(10 + 6\lambda_2) + \frac{12L^4}{t^2} \right. \\
& \quad \left. \left\{ (\lambda_1 + \frac{1}{2}) \frac{1}{R_2^2} + \frac{1}{2R_1^2} + \frac{\mu}{R_1 R_2} \right\} \right] w_{JK} \\
& - 4(1+\lambda_2) w_{JK-1} + (\frac{1}{2} + \lambda_2) w_{JK-2} - (7-\mu) w_{J-1 K} + (2-\frac{\mu}{2}) w_{J-1 K+1} \\
& \quad + (2-\frac{\mu}{2}) w_{J-1 K-1} + w_{J-2 K} \\
& \frac{-12L^3}{t^2} \left\{ \frac{1}{R_2} (\frac{1}{2} + \lambda_1) + \frac{\mu}{2R_1} \right\} (v_{JN} - v_{Jn-1}) \\
& \frac{-12L^3}{t^2} \left\{ \left(\frac{1}{R_1} + \frac{\mu}{R_2} \right) (u_{JK} - u_{m-1 K}) \right\} - \frac{p_n L^4}{2K} = 0 \quad 5.26
\end{aligned}$$

Eqns. 5.24 to 5.26 involve the tangential displacement of the beam and the displacements of fictitious points outside the shell. Two additional equations are

required and these equations are obtained by considering the tangential and moment equilibrium of the deformable element JK as given below.

$$S_{Jn} - S_{Jn-1} - N_{xx})_{JK} = 0 \quad 5.27$$

$$T_{Jn} - T_{Jn-1} - M_{xx})_{JK} = 0 \quad 5.28$$

The shears in the beam in the plane of the shell are related to the displacements by the following expressions.

$$\begin{aligned} S_{Jn} &= \frac{1}{L} \left(M''_{yy})_{JK+1} - M''_{yy})_{JK} \right) \\ &= \frac{K\lambda_4}{L^2} \left(u_{JK+2} - 3 u_{JK+1} + 3 u_{JK} - u_{JK-1} \right) \end{aligned} \quad 5.29$$

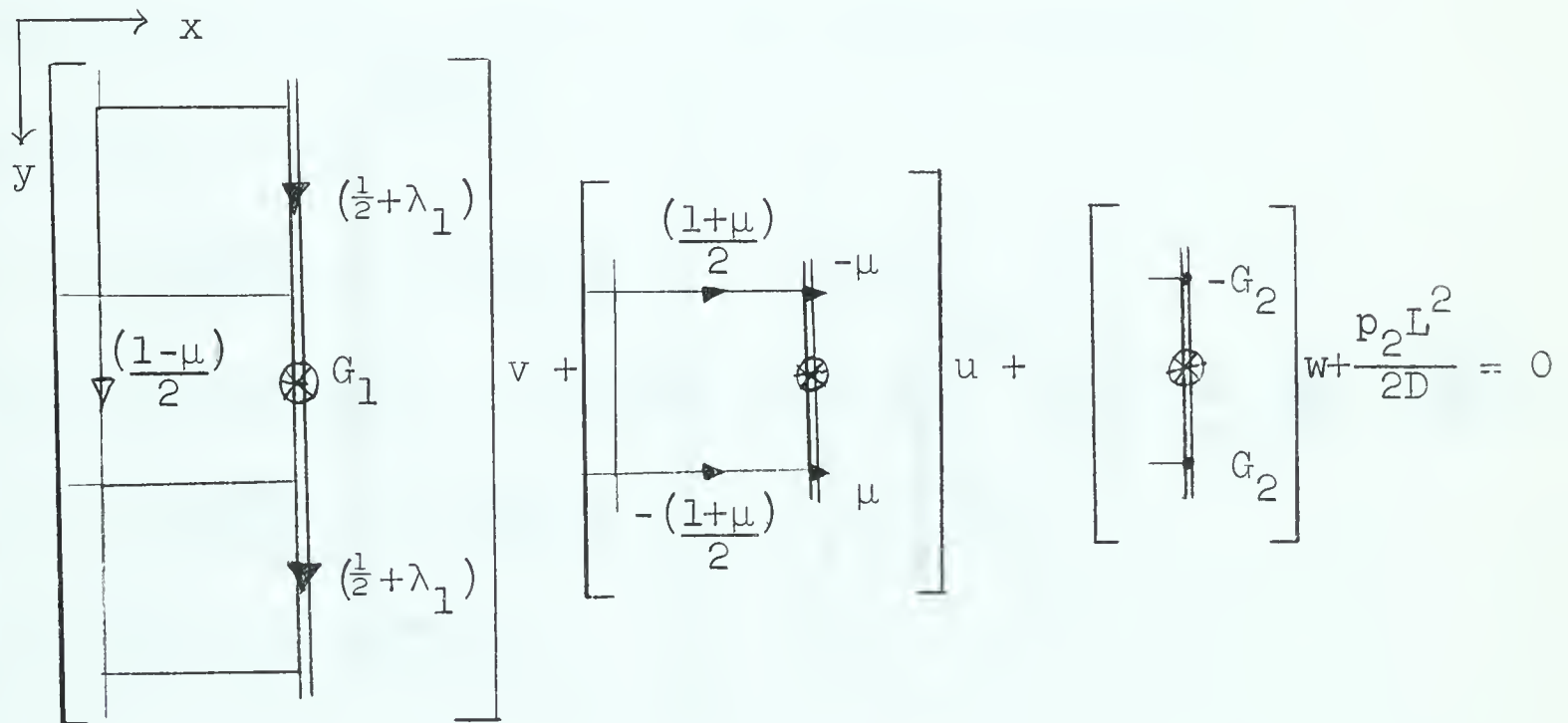
$$S_{Jn-1} = \frac{K\lambda_4}{L^2} \left(u_{JK+1} - 3 u_{JK} + 3 u_{JK-1} - u_{JK-2} \right) \quad 5.30$$

Eqns. 5.27 and 5.28 can be expressed in terms of the displacements as follows.

$$\left\{ \begin{aligned} &\frac{t^2}{12L^2} \lambda_4 (u_{JK+2} - 4 u_{JK+1} - 4 u_{JK-1} + u_{JK-2}) \\ &+ \left(\frac{t^2 \lambda_4}{2L^2} + 2 \right) u_{JK} - 2 u_{m-1} K \\ &+ \mu (v_{Jn} - v_{Jn-1}) - L \left(\frac{1}{R_1} + \frac{\mu}{R_2} \right) w_{JK} = 0 \end{aligned} \right. \quad 5.31$$

$$\begin{aligned}
& \lambda_3 w_{J+1, K+1} - 2(\lambda_3 - 1) w_{J+1, K} + \lambda_3 w_{J+1, K-1} - 2\mu w_{J, K+1} \\
& + 4(1+\mu) w_{J, K} - 2\mu w_{J, K-1} - \lambda_3 w_{J-1, K+1} + 2(\lambda_3 - 1) w_{J-1, K} \\
& - \lambda_3 w_{J-1, K+1} = 0
\end{aligned} \tag{5.32}$$

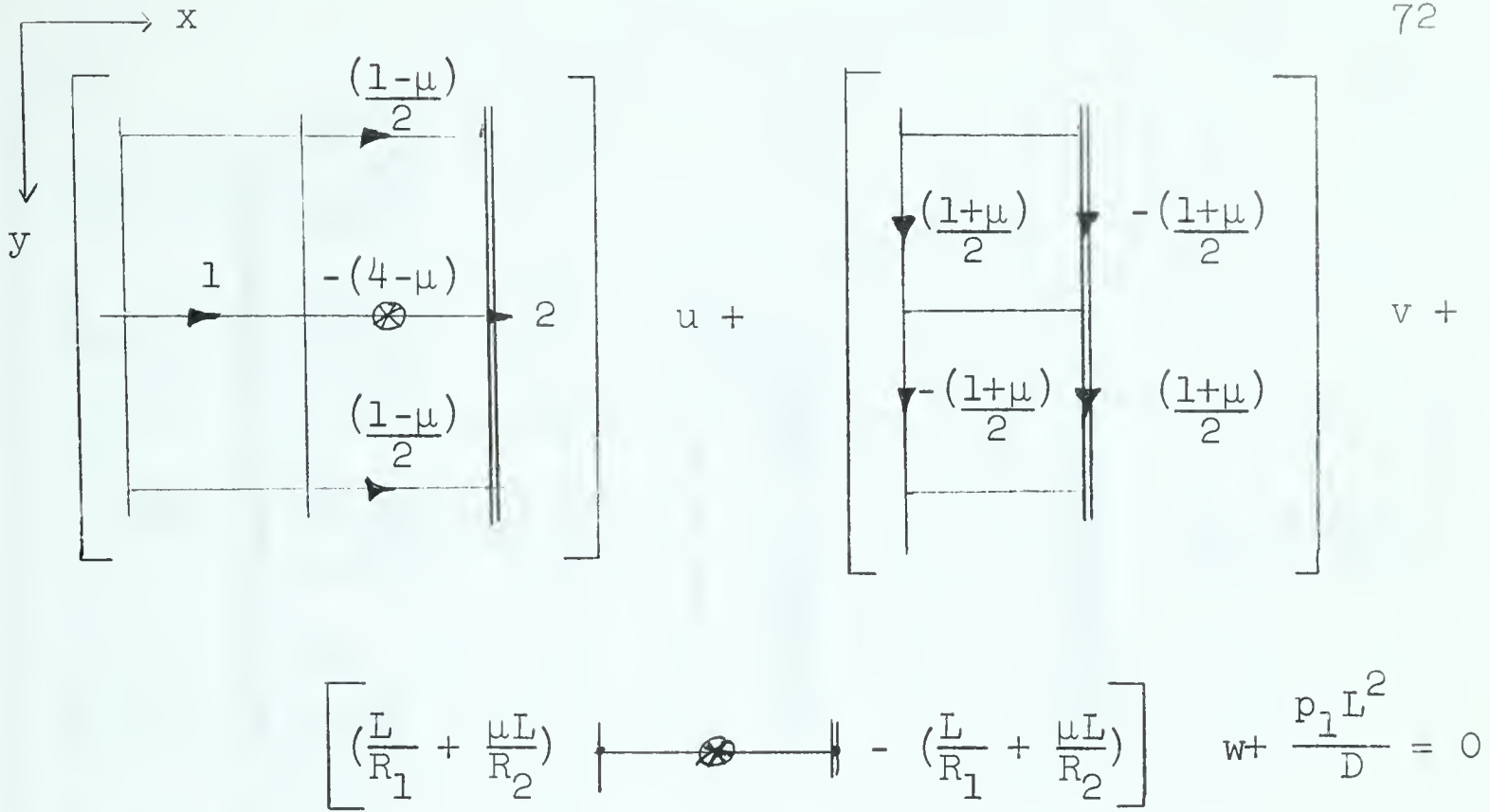
Eqns. 5.31 and 5.32 provide the two additional relationships required. Operator form of Eqns. 5.24 to 5.26 and 5.31 and 5.32 are given below.



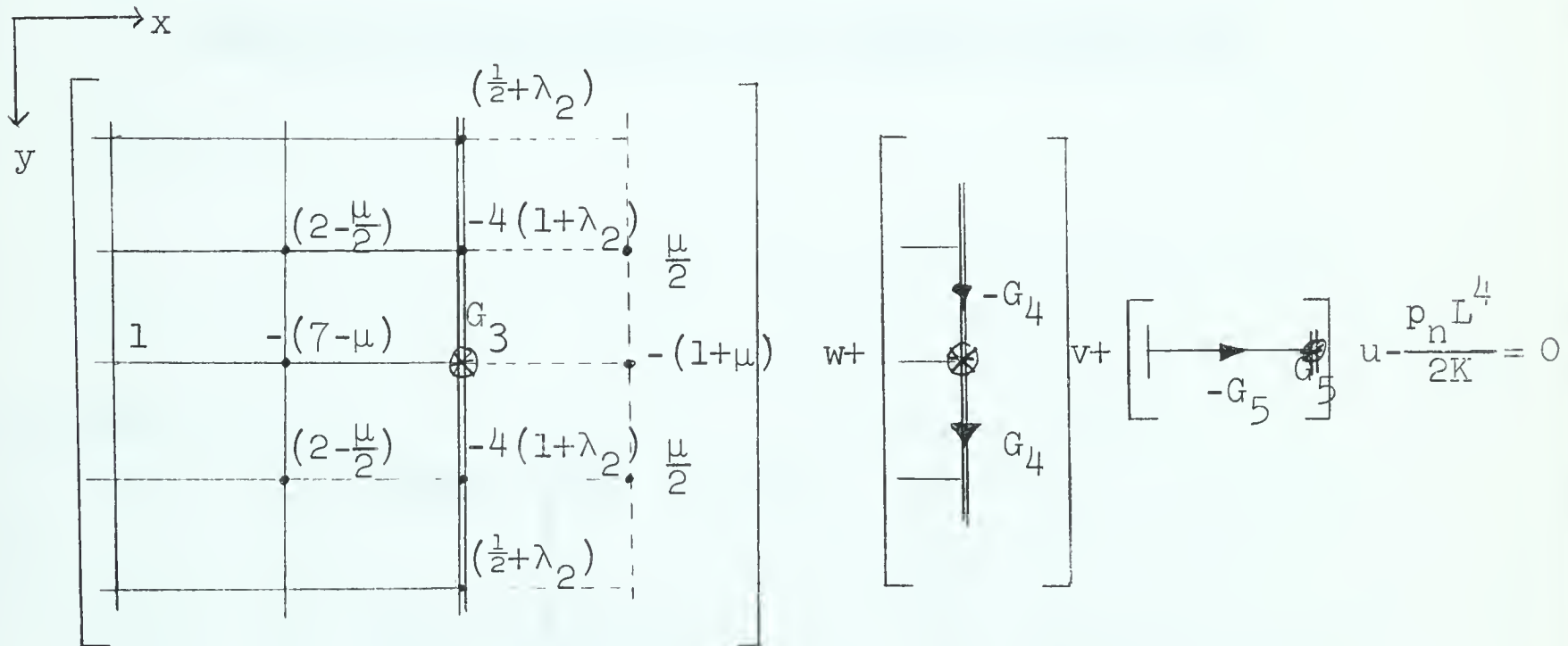
$$G_1 = \left(-\frac{3}{2} - 2\lambda_1 + \frac{\mu}{2} \right)$$

$$G_2 = - \left(\frac{\mu L}{2R_1} + \frac{L}{2R_2} + \frac{L\lambda_1}{R_2} \right) .$$

Equilibrium Equation in the v-direction - Point Jn



Equilibrium Equation in the u -direction - Point $m-1$ K



$$G_3 = \frac{12L^4}{t^2} \left\{ (\lambda_1 + \frac{1}{2}) \frac{1}{R_2} + \frac{1}{2R_1} + \frac{\mu}{R_1 R_2} \right\} + (10 + 6\lambda_2)$$

$$G_4 = - \frac{12L^3}{t^2} \left\{ \frac{\mu}{2R_1} + \frac{1}{R_2} \left(\frac{1}{2} + \lambda_1\right) \right\}$$

$$G_5 = - \frac{12L^3}{t^2} \left(\frac{1}{R_1} + \frac{\mu}{R_2} \right)$$

Equilibrium Equation in the w -direction - Point JK

$$\left[\begin{array}{c} \frac{t^2 \lambda_4}{12 L^2} \\ -\frac{t^2 \lambda_4}{3 L^2} \\ \left(\frac{t^2 \lambda_4}{2 L^2} + 2 \right) \\ \frac{t^2 \lambda_4}{3 L^2} \\ \frac{t^2 \lambda_4}{12 L^2} \end{array} \right] u + \left[\begin{array}{c} -\mu \\ \mu \end{array} \right] v + \left[\begin{array}{c} -\left(\frac{L}{R_1} + \frac{\mu L}{R_2} \right) \end{array} \right] w = 0$$

Tangential Equilibrium of deformable element JK

$$\left[\begin{array}{ccc} -\lambda_3 & -2\mu & \lambda_3 \\ 2(\lambda_3 - 1) & 4(1 + \mu) & -2(\lambda_3 + 1) \\ -\lambda_3 & -2\mu & \lambda_3 \end{array} \right] w = 0$$

Moment equilibrium of deformable element JK

5.6 Summary.

Four types of boundaries were considered in this chapter. For shells with simply supported and fixed edges, additional equations, which relate the displacements of fictitious points directly to the displacements of the real points, were obtained. Using these additional equations, the unknown displacements of the fictitious points can be eliminated from the equilibrium equations for points on or near the boundaries.

The displacements of fictitious points for shells with free edges were eliminated by obtaining the equilibrium equations independently for points on or near the boundaries.

For shells with edges supported on elastic beams, some of the displacements of fictitious points were eliminated by modifying the model near the edges and obtaining the equilibrium equations for points on or near the boundaries. The entire displacements of fictitious points cannot be eliminated from the equilibrium equations in this way. However, the two additional equations obtained in this chapter for this type of boundaries can be used in conjunction with the equilibrium equations to solve for the unknown displacements.

CHAPTER VI

SUMMARY AND CONCLUSIONS

6.1 General Remarks.

Membrane theory, the simplest method of analysing doubly curved shells of translation, due to its many assumptions, is unsatisfactory for shallow shells. Attempts to overcome the limitations of the membrane theory by introducing a bending theory for shallow shells have not been too successful because of the complexity of the differential equations which cannot conveniently be solved, especially for shells supported by elastic beams, a form of support commonly used in practice. In this study, a procedure to account for bending and any boundary conditions for shallow shells is presented.

6.2 Review of the models.

It was found possible to develop physical models to explain the behaviour of the shell structures. The assumptions that are necessary to develop such models are essentially the same as those of the classical bending theories and the degree of accuracy obtained will be similar to that using the finite difference technique to solve the differential equations of these bending theories.

The structures of the physical models A and B developed in Chapter III are the same but the definitions of the displacement systems are different. In model A, the displacement components are defined along the directions parallel to the co-ordinate axes and thus the directions of the displacement components remain the same for the entire shell and this definition is similar to that used by Marguerre. On the other hand, in model B, the displacement components are defined along the tangential and normal directions and thus the directions vary from point to point in the shell and this definition is similar to that used by Vlassov and Conrad. In Chapter IV, it has been shown that there exists a one-to-one correspondence between the equations of the models (A and B) and the finite difference equations of the bending theories of Marguerre and Conrad, respectively. This comparison proves that these physical models can be used for analysing the behaviour of doubly curved shells of translation.

The method of application of the physical-model technique is similar to that of finite difference technique. The shell is divided into a square grid in the projected plane and the equilibrium equations are written at every intersection of the grid lines and at every mid-point of the grid lines. By solving these equations simultaneously,

the displacement components of various points in the shell are obtained from which the stress-resultants can be calculated.

The equations of equilibrium on or near the boundaries involve the displacements of fictitious points outside the shell and this renders the number of equations written for the real points insufficient to solve for the unknowns. This necessitates writing additional equations or modifying the model on the edges and these additional equations and modifications necessary for model B are discussed in Chapter V.

6.3 Range of applicability of the models.

The equations given in Chapters III and V are directly applicable for elliptic paraboloids. When the equations are used for hyperbolic paraboloids, however, it should be noted that the radii of curvatures are of opposite signs and the necessary changes should be made. These equations can also be used for parabolic cylinders by letting one of the curvatures be zero.

In addition to the problems that can be solved by the existing methods, there are many problems that can be solved with the model that cannot be solved by the existing methods. Some examples of the latter type are discussed below.

(i) The model can be used for shells supported on elastic beams where the middle surfaces of the beam and shell coincide. At present, it is very difficult to predict the percentage error that would be introduced if the model is used for shells where the middle surfaces do not coincide. This can only be determined by carrying out model tests with beams at various positions.

(ii) Often shells are built with varying thickness and with beams that have varying dimensions along their length. This method can be used for this type of problem since the stiffnesses of the shell and beams are defined at discrete points in the model.

(iii) The models can be modified for continuous shells. While carrying out the modifications, it should be noted that there is a discontinuity of curvatures at the junction of the shells. A new set of strain-displacement relationships is required at this point.

(iv) The effect of openings in the shells can be studied using this method. The additional equations required adjacent to the openings are obtained in a manner similar to those for the edge depending on whether the openings are stiffened or not.

6.4 Conclusions.

The concept of using a physical model to represent the shell structure provides the design engineer with a procedure to analyse doubly curved shells of translation for any type of boundary condition, loading or internal discontinuity. Using this method as a tool, the effect of various parameters on the behaviour of the shell structure may be studied leading to a greater understanding of the shell behaviour.

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APPENDIX A
DERIVATION OF MARGUERRE'S THEORY

APPENDIX A

DERIVATION OF MARGUERRE'S THEORY

A.1 Introduction

This appendix contains the derivation of the bending theory of shallow shells based on Marguerre's technique. The displacement of a point is defined by three components along the directions parallel to the co-ordinate axes. The forces are projected to the x-y plane to obtain the equilibrium equations. The stress resultants acting on an element are shown in Fig. A.1.

The projected lengths (dx, dy) of an element are given by

$$dx = ds_1 \cos \beta$$

$$dy = ds_2 \cos \alpha$$

The fictitious forces ($\bar{N}_{xx}$, $\bar{N}_{yy}$, $\bar{N}_{xy}$) in the projected plane are related to the corresponding stress resultants (N_{xx} , N_{yy} , N_{xy}) by the following expressions.

$$\begin{aligned} \bar{N}_{xx} &= (N_{xx} ds_2 \cos \beta) \frac{1}{dy} \\ &= N_{xx} \frac{\cos \beta}{\cos \alpha} \end{aligned} \quad \text{A.1}$$

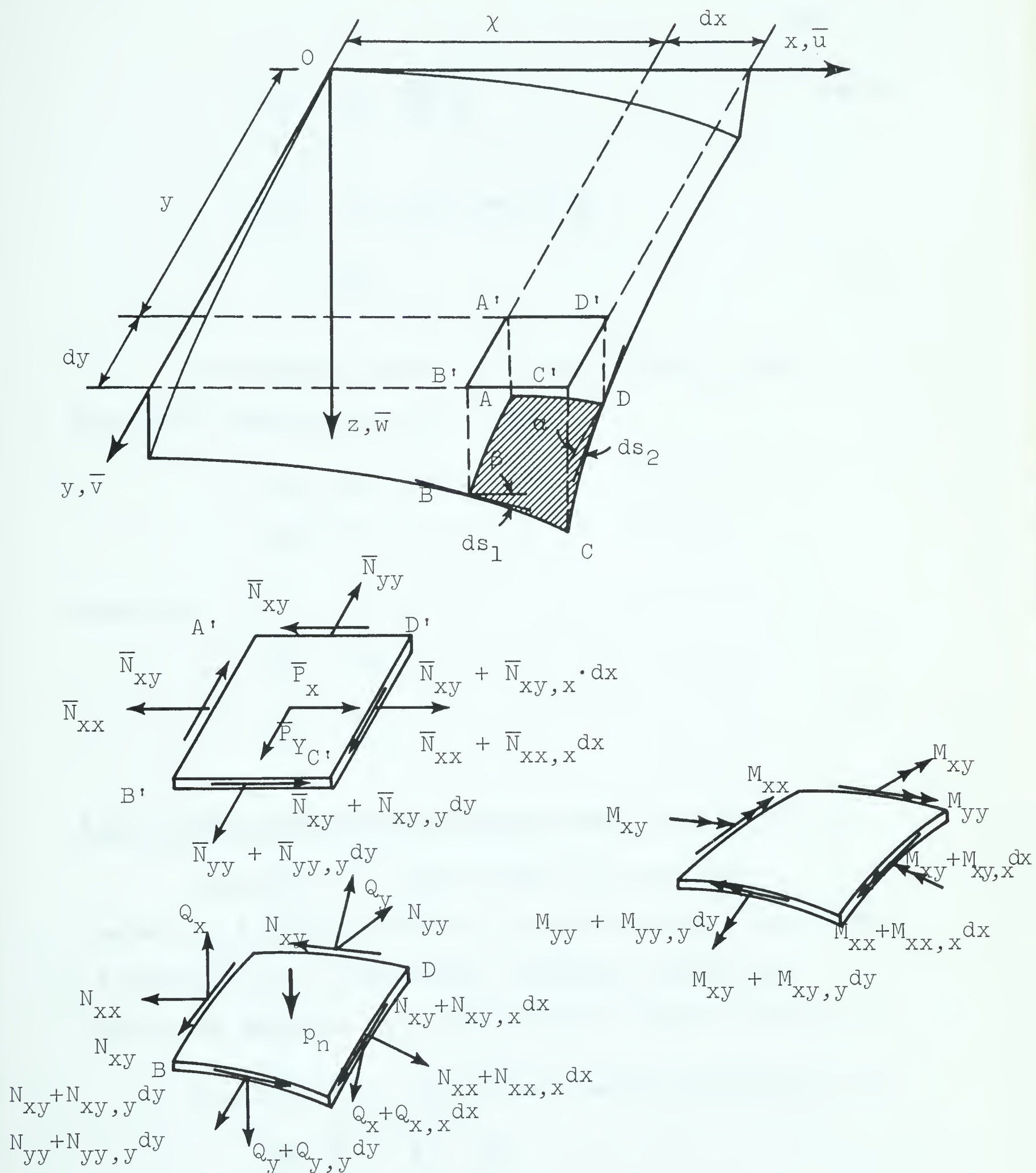


Fig. A.1 STRESS RESULTANTS - MARGUERRE'S THEORY

$$\bar{N}_{yy} = N_{yy} \frac{\cos \alpha}{\cos \beta} \quad A.2$$

$$\begin{aligned} \bar{N}_{xy} &= (N_{xy} ds_1 \cos \beta) \frac{1}{dx} \\ &= N_{xy} \end{aligned} \quad A.3$$

For shallow shells, the squares of $z_{,x}$ and $z_{,y}$ are small compared to unity. Hence,

$$\cos \alpha \doteq 1$$

$$\cos \beta \doteq 1$$

Therefore,

$$\bar{N}_{xx} \doteq N_{xx} \quad A.4$$

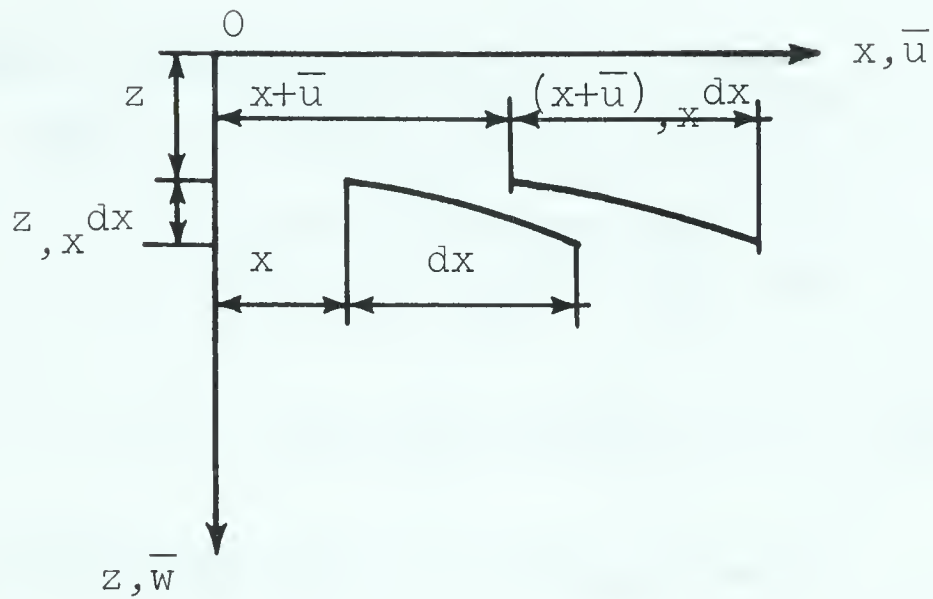
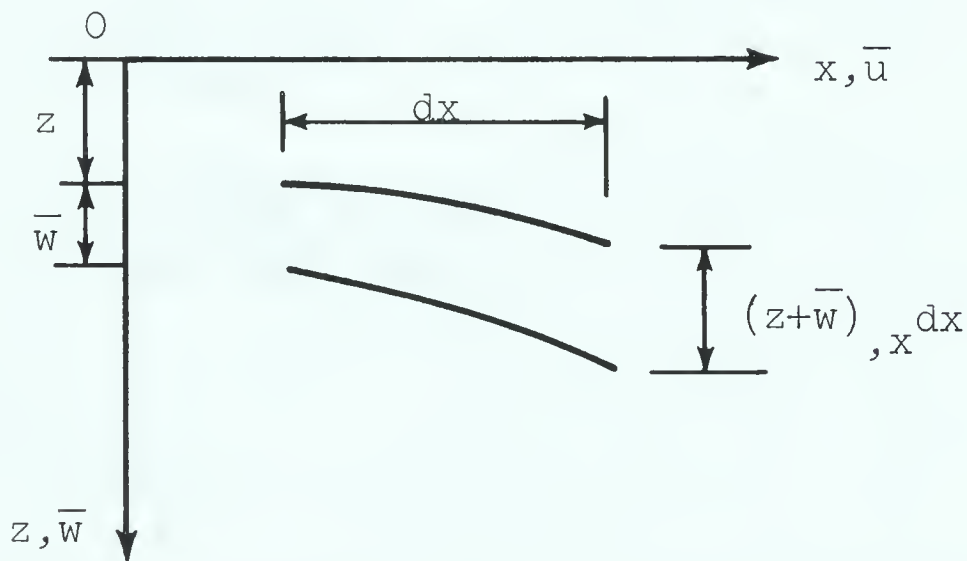
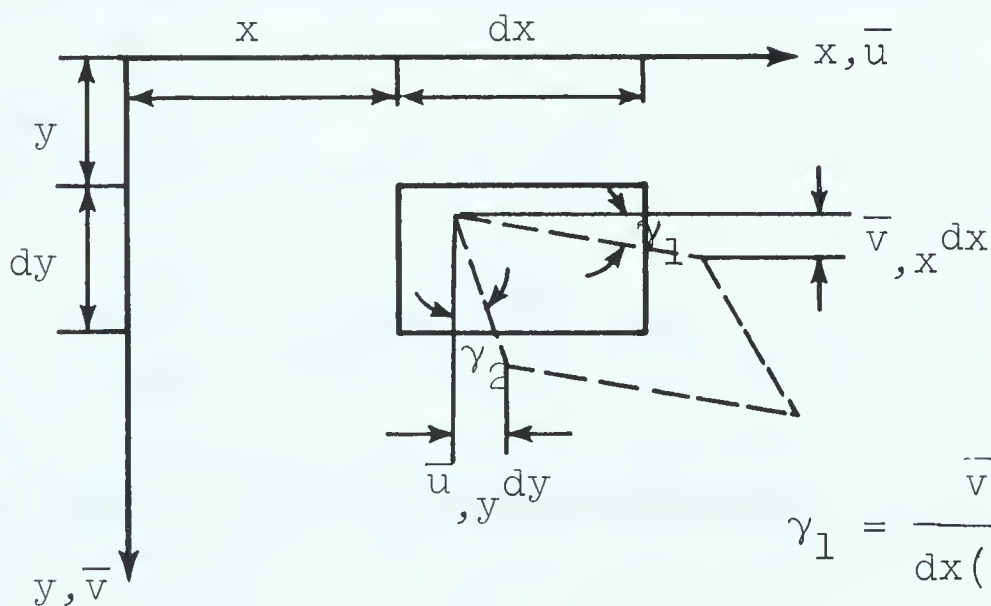
$$\bar{N}_{yy} \doteq N_{yy} \quad A.5$$

A.2 Stress-Displacement Relationships

Consider the normal strain ϵ_x of a line element in a plane parallel to the x-z plane. This strain is caused by the displacement components along the directions parallel to x and z axes as shown in Fig. A.2.

Length of the line element before deformation is

$$\begin{aligned} ds_1 &= \sqrt{1 + z_{,x}^2} dx \\ &\doteq (1 + \frac{1}{2} z_{,x}^2 + \dots) dx \end{aligned}$$

(a) Effect of $\bar{u}$ displacement on Extensional Strain(b) Effect of $\bar{w}$ displacement on Extensional Strain(c) Effect of $\bar{u}$, $\bar{v}$ displacements on shear strain

$$\gamma_1 = \frac{\bar{v}_{,x} dx}{dx(1 + \bar{u}_{,x})} \approx \bar{v}_{,x}$$

$$\gamma_2 = \frac{\bar{u}_{,y} dy}{dy(1 + \bar{v}_{,y})} \approx \bar{u}_{,y}$$

and after $\bar{u}$ displacement is

$$\begin{aligned} ds_1^* &= \sqrt{\{(x+\bar{u})_{,x} dx\}^2 + (z_{,x} dx)^2} \\ &= \sqrt{(1 + \bar{u}_{,x})^2 + z_{,x}^2} dx \\ &\simeq (1 + \bar{u}_{,x} + \frac{1}{2} z_{,x}^2 + \dots) dx \end{aligned}$$

Hence, the strain due to $\bar{u}$ displacement

$$\begin{aligned} &= \frac{ds_1^* - ds_1}{ds_1} \\ &\simeq \frac{\bar{u}_{,x} dx}{(1 + \frac{1}{2} z_{,x}^2) dx} \\ &\simeq \bar{u}_{,x} \end{aligned}$$

Length of the line element after $\bar{w}$ displacement is

$$\begin{aligned} ds_1^{**} &= \sqrt{\{(z+\bar{w})_{,x} dx\}^2 + dx^2} \\ &\simeq \left[1 + \frac{1}{2} \{(z+\bar{w})_{,x}\}^2 \right] dx \end{aligned}$$

Hence, the strain due to $\bar{w}$ displacement is

$$\begin{aligned}
&= \frac{ds_1^{**} - ds_1}{ds_1} \\
&\approx \frac{\frac{1}{2}(\bar{w}_{,x} + z_{,x})^2 - \frac{1}{2}z_{,x}^2}{1 + \frac{1}{2}z_{,x}^2} \\
&\approx z_{,x}\bar{w}_{,x} + \frac{1}{2}\bar{w}_{,x}^2
\end{aligned}$$

The total strain is

$$\epsilon_x = \bar{u}_{,x} + z_{,x}\bar{w}_{,x} + \frac{1}{2}\bar{w}_{,x}^2 \quad A.6$$

By evaluating the order of magnitude of the quantities on the right-hand side of Eqn. A.6, it can be seen that the last term is one order higher than the rest of the terms and it can be neglected. Marguerre retained this term in his original derivation to obtain the non-linear theory, but it was neglected later to obtain the linear theory.

$$\text{Thus,} \quad \epsilon_x = \bar{u}_{,x} + z_{,x}\bar{w}_{,x} \quad A.7$$

$$\text{Similarly,} \quad \epsilon_y = \bar{v}_{,y} + z_{,y}\bar{w}_{,y} \quad A.8$$

The shear strain, γ_{xy} , in the middle surface is defined as the change in angle between the sides of the element of the shell. From Fig. A.2, it is seen that the change in angle due to $\bar{v}$ displacement is

$$\frac{\bar{v}_{,x} dx}{(1+u_{,x}) dx}$$

$$\doteq \bar{v}_{,x}$$

Similarly, due to $\bar{u}$ displacement, the change in angle is $\bar{u}_{,y}$.

The original angle, θ , between the sides of the element is given by

$$\cos \theta = \sin \alpha \sin \beta$$

$$\doteq \tan \alpha \tan \beta$$

$$= z_{,x} z_{,y}$$

After the displacement $\bar{w}$, the angle between the sides of the element is given by

$$\cos (\theta - \gamma_1) = (z + \bar{w})_{,x} (z + \bar{w})_{,y}$$

where γ_1 = shear strain due to $\bar{w}$ displacement.

Since $(\frac{\pi}{2} - \theta)$ is a small angle,

$$\cos \theta = \sin (\frac{\pi}{2} - \theta)$$

$$\doteq \frac{\pi}{2} - \theta$$

and $\cos (\theta-\gamma_1) \doteq \frac{\pi}{2} - \theta + \gamma_1$

Thus,

$$\gamma_1 = z_{,x} \bar{w}_{,y} + z_{,y} \bar{w}_{,x} + \bar{w}_{,x} \bar{w}_{,y}$$

The shear strain = $\gamma_{xy} = \bar{u}_{,y} + \bar{v}_{,x} + z_{,x} \bar{w}_{,y} + z_{,y} \bar{w}_{,x} + \bar{w}_{,x} \bar{w}_{,y}$ A.9

Again, the last term of Eqn. A.9 is small quantity and can be neglected. It is noted that this term was retained by Marguerre in his original derivation but was neglected later on.

Hence, $\gamma_{xy} = \bar{u}_{,y} + \bar{v}_{,x} + z_{,x} \bar{w}_{,y} + z_{,y} \bar{w}_{,x}$ A.10

From Hooke's Law

$$\begin{aligned} N_{xx} &= D (\epsilon_x + \mu \epsilon_y) \\ &= D (\bar{u}_{,x} + \mu \bar{v}_{,y} + z_{,x} \bar{w}_{,x} + \mu z_{,y} \bar{w}_{,y}) \end{aligned} \quad A.11$$

$$\begin{aligned} N_{yy} &= D (\epsilon_y + \mu \epsilon_x) \\ &= D (\bar{v}_{,y} + \mu \bar{u}_{,x} + z_{,y} \bar{w}_{,y} + \mu z_{,x} \bar{w}_{,x}) \end{aligned} \quad A.12$$

$$\begin{aligned} N_{xy} &= \frac{Et}{2(1+\mu)} \gamma_{xy} \\ &= D \left(\frac{1-\mu}{2} \right) (\bar{u}_{,y} + \bar{v}_{,x} + z_{,x} \bar{w}_{,y} + z_{,y} \bar{w}_{,x}) \end{aligned} \quad A.13$$

Neglecting the effect of $\bar{u}$, $\bar{v}$ displacements on the changes in curvatures and twist, the following results are obtained.

$$M_{xx} = -K(\bar{w}_{,xx} + \mu \bar{w}_{,yy}) \quad A.14$$

$$M_{yy} = -K(\bar{w}_{,yy} + \mu \bar{w}_{,xx}) \quad A.15$$

$$M_{xy} = -K(1-\mu) \bar{w}_{,xy} \quad A.16$$

A.3 Equilibrium Equations.

By summing the forces in the x direction and neglecting the effect of Q_x and Q_y ,

$$\bar{N}_{xx,x} + \bar{N}_{xy,y} + \bar{p}_x = 0$$

Substituting the relationships derived in section A.1

$$N_{xx,x} + N_{xy,y} + \bar{p}_x = 0 \quad A.17$$

Similarly

$$N_{yy,y} + N_{xy,x} + \bar{p}_y = 0 \quad A.18$$

Marguerre summed the forces along a direction parallel to the z-axis to obtain his third equilibrium equation. However, the method used by Nash and Sheng [7] is adopted here since it yields a simpler equation. In this method, the forces are summed normal to the element giving,

$$Q_{x,x} + Q_{y,y} + z_{,xx}N_{xx} + z_{,yy}N_{yy} + 2z_{,xy}N_{xy} + p_n = 0 \quad A.19$$

By considering the moment equilibrium of the element about x and y axes,

$$M_{xx,x} + M_{xy,y} = Q_x \quad A.20$$

$$M_{yy,y} + M_{xy,x} = Q_y \quad A.21$$

Substituting Eqns. A.11 to A.16 and Eqns. A.20 and A.21 into Eqns. A.17 to A.19, the following equations which permit the solution of the displacements are obtained .

$$\begin{aligned} \bar{u}_{,xx} + \frac{(1-\mu)}{2}u_{,yy} + \frac{(1+\mu)}{2}\bar{v}_{,xy} + \left\{ z_{,xx} + \frac{(1-\mu)}{2}z_{,yy} \right\} \bar{w}_{,x} \\ + \frac{(1+\mu)}{2}z_{,xy}\bar{w}_{,y} + z_{,x}\bar{w}_{,xx} + \frac{(1-\mu)}{2}z_{,x}\bar{w}_{,yy} \\ + \frac{(1+\mu)}{2}z_{,y}\bar{w}_{,xy} + \frac{\bar{p}_x}{D} = 0 \end{aligned} \quad A.22$$

$$\begin{aligned}
& \bar{v}_{,yy} + \frac{(1-\mu)}{2} \bar{v}_{,xx} + \frac{(1+\mu)}{2} \bar{u}_{,xy} + \left\{ z_{,yy} + \frac{(1-\mu)}{2} z_{,xx} \right\} \bar{w}_{,y} \\
& + \frac{(1+\mu)}{2} z_{,xy} \bar{w}_{,x} + z_{,y} \bar{w}_{,yy} + \frac{(1-\mu)}{2} z_{,y} \bar{w}_{,xx} \\
& + \frac{(1+\mu)}{2} z_{,x} \bar{w}_{,xy} + \frac{p_y}{D} = 0
\end{aligned} \tag{A.23}$$

$$\begin{aligned}
& (\bar{u}_{,x} + z_{,x} \bar{w}_{,x} + \mu \bar{v}_{,y} + \mu z_{,y} \bar{w}_{,y}) z_{,xx} + \\
& (\bar{v}_{,y} + z_{,y} \bar{w}_{,y} + \mu \bar{u}_{,x} + \mu z_{,x} \bar{w}_{,x}) z_{,yy} + \\
& (1-\mu) (\bar{u}_{,y} + \bar{v}_{,x} + z_{,x} \bar{w}_{,y} + z_{,y} \bar{w}_{,x}) z_{,xy} - \\
& \frac{t^2}{12} \nabla^2 \nabla^2 \bar{w} + \frac{p_n}{D} = 0
\end{aligned} \tag{A.24}$$

For shells of the type

$$z = \frac{x^2}{2R_1} + \frac{y^2}{2R_2}$$

the above equations reduce to the following.

$$\begin{aligned}
& \bar{u}_{,xx} + \frac{(1-\mu)}{2} \bar{u}_{,yy} + \frac{(1+\mu)}{2} \bar{v}_{,xy} + \left\{ \frac{1}{R_1} + \frac{(1-\mu)}{2R_2} \right\} \bar{w}_{,x} + \\
& z_{,x} \bar{w}_{,xx} + \frac{(1-\mu)}{2} z_{,x} \bar{w}_{,yy} \\
& + \frac{(1+\mu)}{2} z_{,y} \bar{w}_{,xy} + \frac{\bar{p}_x}{D} = 0
\end{aligned} \tag{A.25}$$

$$\begin{aligned}
& \bar{v}_{,yy} + \frac{(1-\mu)}{2} \bar{v}_{,xx} + \frac{(1+\mu)}{2} \bar{u}_{,xy} + \left\{ \frac{1}{R_2} + \frac{(1-\mu)}{2R_1} \right\} \bar{w}_{,y} + \\
& z_{,y} \bar{w}_{,yy} + \frac{(1-\mu)}{2} z_{,y} \bar{w}_{,xx} + \frac{(1+\mu)}{2} z_{,x} \bar{w}_{,xy} + \frac{p_y}{D} = 0 \quad \text{A.26}
\end{aligned}$$

$$\begin{aligned}
& \left(\frac{1}{R_1} + \frac{\mu}{R_2} \right) \bar{u}_{,x} + \left(\frac{\mu}{R_1} + \frac{1}{R_2} \right) \bar{v}_{,y} + \left(\frac{1}{R_1} + \frac{\mu}{R_2} \right) z_{,x} \bar{w}_{,x} \\
& + \left(\frac{\mu}{R_1} + \frac{1}{R_2} \right) z_{,y} \bar{w}_{,y} - \frac{t^2}{12} \nabla^2 \nabla^2 \bar{w} + \frac{p_n}{D} = 0 \quad \text{A.27}
\end{aligned}$$

Eqns. A.25 to A.27 permit the solution of the displacements from which the stress resultants can be calculated.

APPENDIX B

DERIVATION OF CONRAD'S THEORY

APPENDIX B

DERIVATION OF CONRAD'S THEORY

B.1 Introduction

In this appendix, the equations due to Conrad are derived. Differentiation of the geometry terms expressed in ordinary algebra leads to complex expressions. These are greatly simplified if the derivation is carried out using vector algebra. Thus the middle surface of the undeformed shell can be expressed as follows.

$$\vec{r}(x,y) = \vec{i}x + \vec{j}y + \vec{k}z(x,y) \quad \text{B.1}$$

Vectors, tangent to the shell in the x and y directions, are given by $\vec{r}_{,x}$ and $\vec{r}_{,y}$, respectively, and the normal vector is given by

$$\vec{n} = \frac{\vec{r}_{,x} \times \vec{r}_{,y}}{|\vec{r}_{,x} \times \vec{r}_{,y}|} \quad \text{B.2}$$

Now,

$$\begin{aligned} \vec{r}_{,x} \times \vec{r}_{,y} &= (\vec{i} + \vec{k} z_{,x}) \times (\vec{j} + \vec{k} z_{,y}) \\ &= -\vec{i} z_{,x} - \vec{j} z_{,y} + \vec{k} \\ |\vec{r}_{,x} \times \vec{r}_{,y}| &= \sqrt{1 + z_{,x}^2 + z_{,y}^2} \\ &\doteq 1 \end{aligned}$$

since

B3

$$z_{,x}^2 \ll 1 \quad \text{and} \quad z_{,y}^2 \ll 1 .$$

Therefore,

$$\vec{n} = -\vec{i} z_{,x} - \vec{j} z_{,y} + \vec{k} \quad \text{B.3}$$

Any point in the shell at a distance ξ from the middle surface is given by

$$\begin{aligned} \vec{x}(x,y,z) &= \vec{r}(x,y) + \xi \vec{n}(x,y) \\ &= \vec{i}(x - \xi z_{,x}) + \vec{j}(y - \xi z_{,y}) \\ &\quad + \vec{k}(z + \xi) \end{aligned} \quad \text{B.4}$$

B.2 Deformations.

The displacement of a point in the middle surface is defined by the components u , v and w along the $\vec{r}_{,x}$, $\vec{r}_{,y}$ and $\vec{n}$ directions.

After deformation, the point $\vec{x}$ is at

$$\vec{x} + \delta\vec{x} = \vec{r} + \delta\vec{r} + (\xi + \delta\xi)(\vec{n} + \delta\vec{n}) \quad \text{B.5}$$

The influence of the change in ξ is considered negligible and hence ξ will be treated as constant.

Therefore,

$$\vec{x} + \delta\vec{x} = \vec{r} + \delta\vec{r} + \xi(\vec{n} + \delta\vec{n}) \quad \text{B.6}$$

The increase in $\vec{r}$ in terms of the displacements is

$$\delta \vec{r} = u \frac{\vec{r}_{,x}}{|\vec{r}_{,x}|} + v \frac{\vec{r}_{,y}}{|\vec{r}_{,y}|} + w \vec{n} \quad \text{B.7}$$

Using Eqns. B.1 and B.3, it follows that

$$\delta \vec{r} = \vec{i}(u - w z_{,x}) + \vec{j}(v - w z_{,y}) + \vec{k}(w + u z_{,x} + v z_{,y}) \quad \text{B.8}$$

From Eqn. 2.6, it can be seen that $u z_{,x}$ and $v z_{,y}$ are small order quantities compared to w and can be dropped from Eqn. B.8, leaving

$$\delta \vec{r} = \vec{i}(u - w z_{,x}) + \vec{j}(v - w z_{,y}) + \vec{k} w \quad \text{B.9}$$

Therefore,

$$\vec{r} + \delta \vec{r} = \vec{i}(x + u - w z_{,x}) + \vec{j}(y + v - w z_{,y}) + \vec{k}(z + w) \quad \text{B.10}$$

After deformation, the normal vector may be expressed as

$$\begin{aligned} \vec{n} + \delta \vec{n} &= \frac{(\vec{r} + \delta \vec{r})_{,x} \times (\vec{r} + \delta \vec{r})_{,y}}{|(\vec{r} + \delta \vec{r})_{,x} \times (\vec{r} + \delta \vec{r})_{,y}|} \\ &\approx (\vec{r} + \delta \vec{r})_{,x} \times (\vec{r} + \delta \vec{r})_{,y} \quad \text{B.11} \end{aligned}$$

Substituting Eqn. B.10 into Eqn. B.11 and multiplying the vectors,

$$\begin{aligned}
\vec{n} + \delta\vec{n} = & \vec{i} [(v - w z_{,y})_{,x} (z + w)_{,y} - \{1 + (v - w z_{,y})_{,y}\} \\
& (z + w)_{,x}] \\
& + \vec{j} [(u - w z_{,x})_{,y} (z + w)_{,x} - \{1 + (u - w z_{,x})_{,x}\} \\
& (z + w)_{,y}] \\
& + \vec{k} [\{1 + (u - w z_{,x})_{,x}\} \{1 + (v - w z_{,y})_{,y}\} - \\
& (u - w z_{,x})_{,y} (v - w z_{,y})_{,x}]
\end{aligned} \tag{B.12}$$

Eqn. B.12 can be simplified by retaining terms up to the order of $w \delta$ and neglecting terms of order $w \delta^2$. Thus, the deformed normal is given by

$$\vec{n} + \delta\vec{n} = \vec{i} (z + w)_{,x} + \vec{j} (z + w)_{,y} + \vec{k} \tag{B.13}$$

The length of a line element in the x-direction before deformation is given by

$$\bar{x}_{,x} dx = \{\vec{i} (1 - \xi z_{,xx}) - \vec{j} \xi z_{,xy} + \vec{k} z_{,x}\} dx \tag{B.14}$$

and after deformation by,

$$\begin{aligned}
(\vec{x} + \delta\vec{x})_{,x} dx &= \{(\vec{r} + \delta\vec{r})_{,x} + \xi (\vec{n} + \delta\vec{n})_{,x}\} dx \\
&= \vec{i} \{1 + (u - wz_{,x})_{,x} - \xi(z + w)_{,xx}\} dx + \\
&\quad \vec{j} \{1 + (v - wz_{,y})_{,x} - \xi(z + w)_{,xy}\} dx + \\
&\quad \vec{k} (z + w)_{,x} dx
\end{aligned} \tag{B.15}$$

Neglecting terms of order δ^4 in comparison to terms of order δ^3 , the square of the line element before deformation is

$$ds^2 = (1 - \xi z_{,xx} + z_{,x}^2) dx^2 \tag{B.16}$$

and after deformation is

$$\begin{aligned}
(ds + \delta ds)^2 &= \{1 + 2(u - wz_{,x})_{,x} - 2\xi(z + w)_{,xx} \\
&\quad + z_{,x}^2 + 2z_{,x}w_{,x}\} dx^2
\end{aligned} \tag{B.17}$$

The normal strain ϵ_x is defined by $\frac{\delta ds}{ds}$.
Neglecting $(\delta ds)^2$,

$$(ds + \delta ds)^2 - ds^2 = 2ds\delta ds = 2 ds^2 \epsilon_x \tag{B.18}$$

Substituting Eqns. B.16 and B.17 into Eqn. B.18 and differentiating,

B7

$$2ds^2\epsilon_x = (2u_{,x} - 2wz_{,xx} - 2\xi w_{,xx}) dx^2$$

and hence

$$\epsilon_x = u_{,x} - z_{,xx}w - \xi w_{,xx} \quad \text{B.19(a)}$$

Similarly,

$$\epsilon_y = v_{,y} - z_{,yy}w - \xi w_{,yy} \quad \text{B.19(b)}$$

The shear strain, γ_{xy} , is defined as the decrease in angle between the vectors $\vec{x}_{,x}$ and $\vec{x}_{,y}$. The angle, θ , between the vectors before deformation is given by

$$\cos \theta = \frac{\vec{x}_{,x} \cdot \vec{x}_{,y}}{|\vec{x}_{,x}| |\vec{x}_{,y}|}$$

$$\doteq \vec{x}_{,x} \cdot \vec{x}_{,y}$$

$$= \{\vec{i}(1 - \xi z_{,xx}) - \vec{j} \xi z_{,xy} + \vec{k} z_{,x}\} \cdot$$

$$\{-\vec{i} \xi z_{,xy} + \vec{j}(1 - \xi z_{,yy}) + \vec{k} z_{,y}\}$$

$$\doteq z_{,x} z_{,y} - 2\xi z_{,xy} \quad \text{B.20}$$

and after deformation by,

$$\begin{aligned}
\cos(\theta - \gamma_{xy}) &\doteq (\vec{x} + \delta\vec{x})_{,x} \cdot (\vec{x} + \delta\vec{x})_{,y} \\
&= \{1 + (u - wz_{,x})_{,x} - \xi(z + w)_{,xx}\} \\
&\quad \{(u - wz_{,x})_{,y} - \xi(z + w)_{,xy}\} + \\
&\quad \{(v - z_{,y}w)_{,x} - \xi(z + w)_{,xy}\} \\
&\quad \{1 + (v - wz_{,y})_{,y} - \xi(z + w)_{,yy}\} + \\
&\quad (z + w)_{,x}(z + w)_{,y} \\
&\doteq u_{,y} + v_{,x} - 2wz_{,xy} - 2\xi w_{,xy} \quad \text{B.21} \\
&\quad + z_{,x}z_{,y} - 2\xi z_{,xy}
\end{aligned}$$

Note that the terms of order δ^4 have been neglected in the above equation.

Using the identity,

$$\cos(\theta - \gamma_{xy}) = \cos \theta \cos \gamma_{xy} + \sin \theta \sin \gamma_{xy}$$

and the approximations,

$$\sin \theta = \sqrt{1 - \cos^2 \theta} \doteq 1$$

$$\cos \gamma_{xy} \doteq 1, \quad \sin \gamma_{xy} \doteq \gamma_{xy}$$

one obtains that

$$\begin{aligned}\cos(\theta - \gamma_{xy}) &= \cos \theta + \gamma_{xy} \\ &= z_{,x} z_{,y} - 2\xi z_{,xy} + \gamma_{xy}\end{aligned}$$

Therefore,

$$\gamma_{xy} = u_{,y} + v_{,x} - 2 z_{,xy} w - 2\xi w_{,xy} \quad \text{B.22}$$

Note that the order of quantities retained in the Eqns. B.19 and B.22 are of the order of δ^3 .

B.3 Stress-Strain Relationships.

Assuming plane state of stress on all surfaces parallel to the middle surface and using Hooke's Law, the strains are related to the stresses as follows.

$$\left. \begin{aligned} E \epsilon_1 &= \sigma_1 - \mu \sigma_2 \\ E \epsilon_2 &= \sigma_2 - \mu \sigma_1 \\ \frac{E}{2(1+\mu)} \gamma_{12} &= \tau_{12} \end{aligned} \right\} \quad \text{B.23}$$

Where 1 and 2 denote two perpendicular directions. For shallow shells, θ is very close to a right angle and the error introduced by assuming it as a right angle will be small. Hence,

$$\left. \begin{aligned} \sigma_x &= \frac{E}{1-\mu} (\epsilon_x + \mu \epsilon_y) \\ \sigma_y &= \frac{E}{1-\mu} (\epsilon_y + \mu \epsilon_x) \\ \tau_{xy} &= \frac{E}{2(1+\mu)} \gamma_{xy} \end{aligned} \right\} \quad \text{B.24}$$

These stresses act at a distance ξ from the middle surface and the total forces are obtained by summation.

$$\begin{aligned} N_{xx} &= \int_{-t/2}^{t/2} \sigma_x d\xi & N_{yy} &= \int_{-t/2}^{t/2} \sigma_y d\xi \\ N_{xy} &= \int_{-t/2}^{t/2} \tau_{xy} d\xi & & \\ M_{xx} &= \int_{-t/2}^{t/2} \xi \sigma_x d\xi & M_{yy} &= \int_{-t/2}^{t/2} \xi \sigma_y d\xi \\ M_{xy} &= \int_{-t/2}^{t/2} \xi \tau_{xy} d\xi & & \end{aligned} \quad \text{B.25}$$

On integration, the following relationships are obtained

$$\begin{aligned} N_{xx} &= D \{ u_{,x} + \mu v_{,y} - (z_{,xx} + \mu z_{,yy}) w \} \\ N_{yy} &= D \{ v_{,y} + \mu u_{,x} - (z_{,yy} + \mu z_{,xx}) w \} \\ N_{xy} &= D \frac{(1-\mu)}{2} \{ u_{,y} + v_{,x} - 2 z_{,xy} w \} \end{aligned} \quad \text{B.26}$$

(continued next page)

$$M_{xx} = -k (w_{,xx} + \mu w_{,yy})$$

$$M_{yy} = -k (w_{,yy} + \mu w_{,xx})$$

$$M_{xy} = -k (1-\mu) w_{,xy} \quad B.26$$

B.4 Equilibrium.

Transverse shear resultants enter into the equilibrium equations and can be defined as

$$Q_x = \int_{-t/2}^{t/2} \tau_{xz} d\xi$$

B.27

$$Q_y = \int_{-t/2}^{t/2} \tau_{yz} d\xi$$

The stress resultants and the external loads acting on an element of the shell is shown in Fig. B.1. For shallow shells, the element can be considered as a rectangle with sides dx and dy .

By summing the forces in the tangential directions and neglecting the changes in Q_x and Q_y ,

$$N_{xx,x} + N_{xy,y} - z_{,xx} Q_x - z_{,xy} Q_y + p_1 = 0 \quad B.28(a)$$

$$N_{yy,y} + N_{xy,x} - z_{,yy} Q_y - z_{,xy} Q_x + p_2 = 0 \quad B.28(b)$$

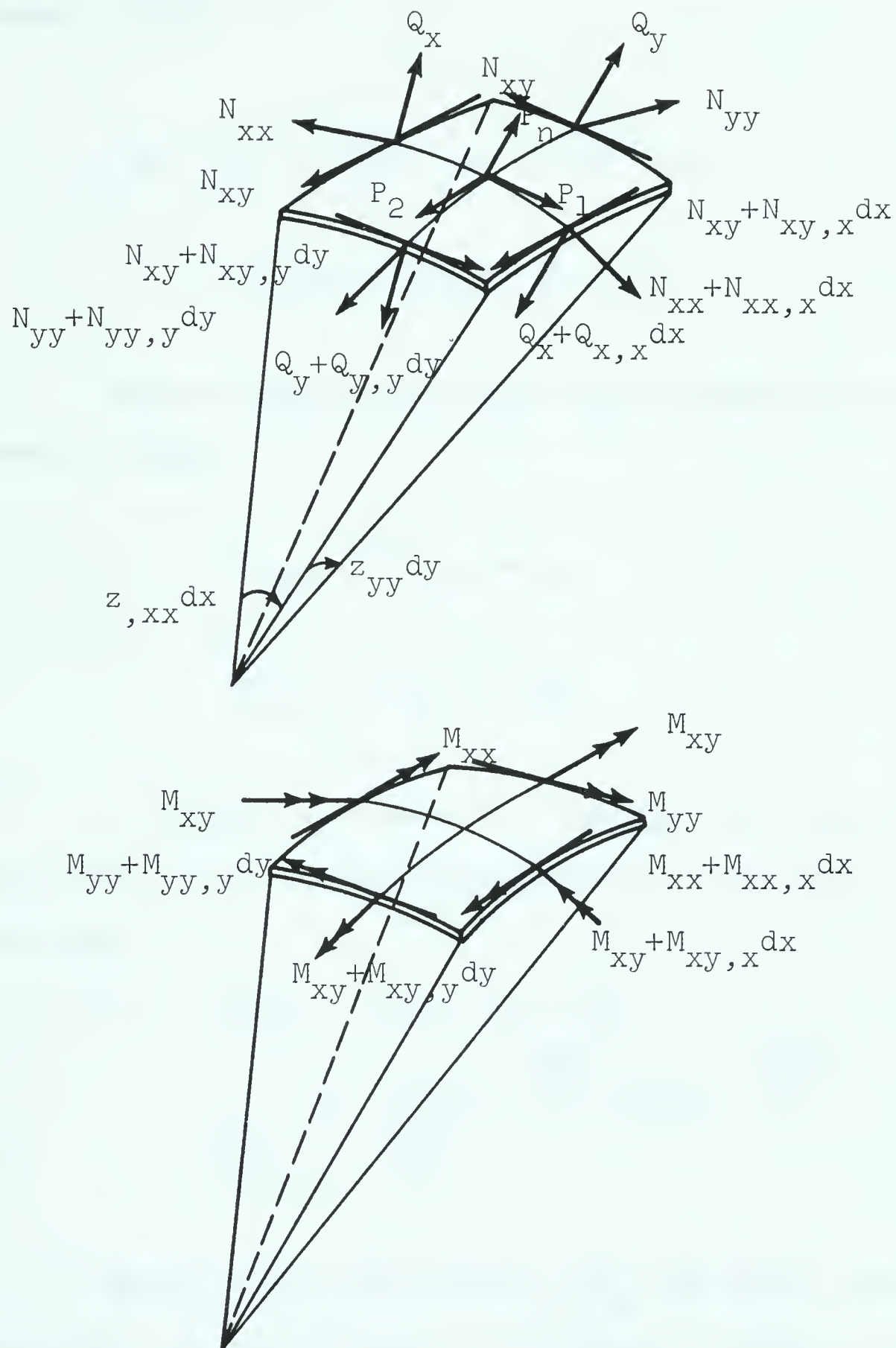


Fig. B.1 STRESS RESULTANTS - CONRAD'S THEORY

Summation of forces along the normal to the element yields

$$N_{xx} z_{,xx} + 2 N_{xy} z_{,xy} + N_{yy} z_{,yy} + Q_{x,x} + Q_{y,y} + p_n = 0 \quad B.29$$

Moment equilibrium about axes tangential to the element yields

$$M_{xx,x} + M_{xy,y} = Q_x \quad B.30(a)$$

$$M_{yy,y} + M_{xy,x} = Q_y \quad B.30(b)$$

It is now necessary to evaluate the order of magnitudes of the various quantities in the above equations.

$$\begin{aligned} N_{xx,x} &\sim Et u_{,xx} \sim E\delta^4 \\ Q_x &\sim M_{xx,x} \sim \frac{Et^3}{12} w_{,xxx} \sim \frac{Et\delta^4}{12} \\ \text{and} \quad Q_x z_{,xx} &\sim \frac{E\delta^6}{12} \end{aligned}$$

Hence terms involving $z_{,xx} Q_x$ and other terms of the same order of magnitude can be dropped from Eqn. B.28 leaving

$$N_{xx,x} + N_{xy,y} + p_1 = 0 \quad \text{B.31(a)}$$

$$N_{yy,y} + N_{xy,x} + p_2 = 0 \quad \text{B.31(b)}$$

Eqn. B.29 can be written as

$$M_{xx,xx} + 2 M_{xy,xy} + M_{yy,yy} + \\ N_{xx}z_{,xx} + 2 N_{xy}z_{,xy} + N_{yy}z_{,yy} + p_n = 0 \quad \text{B.32}$$

By substituting Eqn. B.26 into Eqns. B.31 and B.32 and replacing $z_{,xx}$ by k_1 , $z_{,yy}$ by k_2 and $z_{,xy}$ by k_{12} the following equations are obtained.

$$u_{,xx} + \frac{(1-\mu)}{2} u_{,yy} + \frac{(1+\mu)}{2} v_{,xy} - \{(k_1 + \mu k_2)w\}_{,x} - \\ \{(1-\mu)k_{12}w\}_{,y} + \frac{p_1}{D} = 0 \quad \text{B.33}$$

$$v_{,yy} + \frac{(1-\mu)}{2} v_{,xx} + \frac{(1+\mu)}{2} u_{,xy} - \{(k_2 + \mu k_1)w\}_{,y} - \\ \{(1-\mu)k_{12}w\}_{,x} + \frac{p_2}{D} = 0 \quad \text{B.34}$$

$$(k_1 + \mu k_2)u_{,x} + (k_2 + \mu k_1)v_{,y} + k_{12}(1-\mu)(u_{,y} + v_{,x}) - \\ \{k_1^2 + 2\mu k_1 k_2 + k_2^2 + 2(1-\mu)k_{12}\}w - \frac{t^2}{12} \nabla^2 \nabla^2 w + \frac{p_n}{D} = 0$$

For shells of the form,

$$z = \frac{x^2}{2R_1} + \frac{y^2}{2R_2} ,$$

the third partial derivatives and the mixed derivatives are zero and the equations can be simplified as follows.

$$u_{,xx} + \frac{(1-\mu)}{2} u_{,yy} + \frac{(1+\mu)}{2} v_{,xy} - \left(\frac{1}{R_1} + \frac{\mu}{R_2}\right) w_{,x} + \frac{p_1}{D} = 0 \quad \text{B.36}$$

$$v_{,yy} + \frac{(1-\mu)}{2} v_{,xx} + \frac{(1+\mu)}{2} u_{,xy} - \left(\frac{1}{R_2} + \frac{\mu}{R_1}\right) w_{,y} + \frac{p_2}{D} = 0 \quad \text{B.37}$$

$$\begin{aligned} \left(\frac{1}{R_1} + \frac{\mu}{R_2}\right) u_{,x} + \left(\frac{\mu}{R_1} + \frac{1}{R_2}\right) v_{,xy} - \left(\frac{1}{R_1^2} + \frac{2\mu}{R_1 R_2} + \frac{1}{R_2^2}\right) w \\ - \frac{t^2}{12} \nabla^2 \nabla^2 w + \frac{p_n}{D} = 0 \end{aligned} \quad \text{B.38}$$

Eqns. B.36 to B.38 express the conditions of equilibrium in terms of the displacements. By determining the displacements, the stress-resultants can be obtained.

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